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Short Communication

A new proof of Nguyen's compatibility theorem in a more general context

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Abstract

Let $f(\widetilde{R},\widetilde{S})$ be the image of a pair of fuzzy subsets constructed by applying Zadeh's (1975) extension principle to a function of two variables. Nguyen (1978) gave a necessary and sufficient condition for the α-cuts of $f(\widetilde{R},\widetilde{S})$ to be equal to the crisp images of the α-cuts of $\widetilde{R},\widetilde{S}$. Here we give a simplified proof of this theorem which also holds in a more general context: particularly for second-order fuzzy subsets. © 1998 Elsevier Science B.V.

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1. The compatibility result

Let $(L, \preceq)$ be a complete lattice with minimum and maximum elements denoted respectively by m and M, and let $\widetilde{\mathscr{P}}_L(X)$ be the family of L-fuzzy subsets of the space X, that is the family of maps $(\widetilde{A})$ from X to L. For each $l \in L$ the l-cut of $\widetilde{A}$ is the crisp subset $A_l = \{x \in X \mid \widetilde{A}(x) \succeq l\}$.

Definition 1. Let $\{A_l^* \mid l \in L\}$ be a nested family of crisp subsets of X (in the sense that $l', l'' \in L, l' \prec l'' \Rightarrow A_{l'}^* \supseteq A_{l''}^*$). We say that $\{A_l^*\}$ generates (is a generator of) the fuzzy subset $\widetilde{A}$ if

$$\widetilde{A}(x) = \sup\{l \mid x \in A_l^*\}. \tag{1}$$

Proposition 1. *It is evident that the class $\{A_l\}$ of the l-cuts is a (canonical) generator of $\widetilde{A}$, and moreover, if $\{A_l^*\}$ is another generator of $\widetilde{A}$, then $A_l^* \subseteq A_l$.*

In fact if $x \in A_t^*$ then $t \preceq \sup\{l \mid x \in A_l^*\} = \widetilde{A}(x)$, and therefore $x \in A_t$.

Proposition 2. *A necessary and sufficient condition for $A_l^* = A_l$ is the following:*

$$\sup\{l \mid x \in A_l^*\} = \max\{l \mid x \in A_l^*\}. \tag{2}$$

(a) (*Necessity*) $A_l^* = A_l \Rightarrow \sup = \max$. Suppose $\widetilde{A}(x) = t$. Then $x \in A_t$ and therefore $x \in A_t^*$. Since $x \in A_t^*$ we have $t \preceq \widetilde{A}(x)$, and since $\widetilde{A}(x) = t$ we have $t = \max\{l \mid x \in A_l^*\}$. Thus the necessity of the condition is proved.

(b) (*Sufficency*) $\sup = \max \Rightarrow A_t^* = A_t$. We already know that $A_t^* \subseteq A_t$. Now we will prove that $A_t \subseteq A_t^*$. Suppose $x \in A_t$; then $\widetilde{A}(x) = p \succeq t$. But $\widetilde{A}(x) = \sup\{l \mid x \in A_l^*\} = \max\{l \mid x \in A_l^*\}$ (by assumption), and therefore $x \in A_p^*$. Since $A_p^* \subseteq A_t^*$ (because $p \succeq t$) we also have $x \in A_t^*$ and then the sufficiency of the condition is proved.

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Let f be a map from X to Y and let $f(\widetilde{A})$ be the L-fuzzy set induced on Y by $\widetilde{A}$, via Zadeh's extension principle [2], that is

$$f(\widetilde{A})(y) = \begin{cases} \sup I(y) & \text{if } y \in f(X), \\ m & \text{otherwise,} \end{cases} \tag{3}$$

where $I(y) = \{\widetilde{A}(x) \mid f(x) = y\}$.

Proposition 3. *The family $\{f(A_l)\}$ of the images of the l-cuts is a generator of $f(\widetilde{A})$.*

In order to prove this result let $\beta(y)$ and $\gamma(y)$ be defined as follows:

$$\beta(y) = f(\widetilde{A})(y), \tag{4}$$

$$\gamma(y) = \begin{cases} \sup R(y) & \text{if } R(y) \neq \emptyset, \\ m & \text{otherwise.} \end{cases} \tag{5}$$

where $R(y) = \{l \in L \mid y \in f(A_l)\}$. We have to prove that

$$\beta(y) = \gamma(y). \tag{6}$$

Since $A_m = X$ it is easy to check that $y \notin f(X)$ iff $R(y) = \emptyset$. In fact, if $y \notin f(X) = f(A_m)$, then $y \notin f(A_l)\ \forall l$ (because $A_l \subseteq X$); therefore $R(y) = \{l \mid y \in f(A_l)\} = \emptyset$. On the other hand, if $R(y) = \emptyset$, then $y \notin f(A_l)\ \forall l$, and in particular $y \notin f(A_m) = f(X)$. Thus equality (6) holds in this case. What we have to do is then to prove that, when $y \in f(X)$ and $R(y) \neq \emptyset$, we have

- $\{f(A_l)\}$ is a nested family in the sense of Definition 1 [this is quite evident because $l \preceq n \Rightarrow A_l \supseteq A_n \Rightarrow f(A_l) \supseteq f(A_n)$],
- Eq. (6) holds.

(a) $\gamma(y) \preceq \beta(y)$. If $\alpha \in R(y)$, then by definition $y \in f(A_\alpha)$. Therefore $\exists\, \bar{x} \in A_\alpha$ such that $f(\bar{x}) = y$ and $\widetilde{A}(\bar{x}) \succeq \alpha$ (because $\bar{x} \in A_\alpha$). Then we have

$$\alpha \preceq \widetilde{A}(\bar{x}) \preceq \sup\{\widetilde{A}(x) \mid f(x) = y\}$$

$$= \sup I(y) = \beta(y).$$

So we proved that $\beta(y)$ is larger than all values $\alpha \in R(y)$. Therefore $\beta(y) \succeq \sup R(y) = \gamma(y)$.

(b) $\gamma(y) \succeq \beta(y)$. Let us consider an element $t \in I(y)$. By definition there exists a point $x^* \in X$ such that $f(x^*) = y$ and $\widetilde{A}(x^*) = t$ that is $x^* \in A_t$. This means that $y \in f(A_t)$. Therefore $t \in R(y)$. So we proved that $I(y) \subseteq R(y)$ and consequently

$$\beta(y) = \sup I(y) \preceq \sup R(y) = \gamma(y).$$

Clearly (a) and (b) imply equality (6).

Corollary 1. $\sup\{\widetilde{A}(x) \mid f(x) = y\} = \max\{\widetilde{A}(x) \mid f(x) = y\}$ *is a necessary and sufficient condition in order to have* $[f(\widetilde{A})]_l = f(A_l)$.

The proof is an immediate consequence of Propositions 2 and 3. In fact, we can use Proposition 3 to deduce that $\{f(A_l)\}$ is a generator of $f(\widetilde{A})$. Thus, by using Proposition 2, we obtain the thesis.

Corollary 2 (Nguyen's result). *If $f(u,v)$ is a function of two variables defined on $U \times V$ and $\widetilde{R}, \widetilde{S}$ are two fuzzy subsets of U and V, then we have*

$$\begin{aligned} &[f(\widetilde{R},\widetilde{S})]_l = f(R_l, S_l) \\ &\iff \sup\{\min[\widetilde{R}(u), \widetilde{S}(v)] \mid f(u,v) = y\} \\ &= \max\{\min[\widetilde{R}(u), \widetilde{S}(v)] \mid f(u,v) = y\}. \end{aligned} \tag{7}$$

In order to prove this result, it is sufficient to apply Corollary 1 to the case where $X = U \times V, \widetilde{A} = \widetilde{R} \times \widetilde{S}$, with $\widetilde{R} \in \mathscr{P}(U), \widetilde{S} \in \mathscr{P}(V)$ and $(\widetilde{R} \times \widetilde{S})(u,v) = \min[\widetilde{R}(u), \widetilde{S}(v)]$. It is easy to recognize that $(\widetilde{R} \times \widetilde{S})_l = (R_l \times S_l)$ $[\min\{\widetilde{R}(u), \widetilde{S}(v)\} \succeq l \Leftrightarrow \widetilde{R}(u) \succeq l, \widetilde{S}(v) \succeq l]$. Nguyen's theorem follows immediatly.

It is evident that the same result also holds if we apply the extension principle to a function of several variables, i.e. if $X = U_1 \times U_2 \times \cdots \times U_n$.

Corollary 3. *Nguyen's compatibility result also holds for second-order fuzzy sets.*

This descends from the fact that the family L of the functions from $[0,1]$ to $[0,1]$, equipped with the order relation $f \preceq g \Leftrightarrow f(x) \leqslant g(x) \forall x \in [0,1]$, is a complete lattice.

2. Two examples

We will show here two examples which refer to Corollary 1. We will point out that the existence or the absence of the condition "sup"="max" may depend either on the subset $\widetilde{A}$ or on the function f.

Example 1. Let $(L = [0,1], \preceq)$ be the lattice defined by

- $t \preceq 1$, $0 \preceq t$,
- if x, y are rational, then $x \preceq y \Longleftrightarrow x \leqslant y$,
- if x, y are irrational, then $x \preceq y \Longleftrightarrow x \leqslant y$,
- if x is rational and y is irrational (or vice-versa), then x and y are not comparable.

It is easy to check that

$$x \vee y = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are not comparable,} \\ \max(x,y) & \text{otherwise,} \end{cases}$$

$$x \wedge y = \begin{cases} 0 & \text{if } x \text{ and } y \text{ are not comparable,} \\ \min(x,y) & \text{otherwise} \end{cases}$$

In this example both spaces X and Y are the interval $[0,1]$ and the subset $\tilde{A}$ is given by $\tilde{A}(x) = x$. Note the difference between the two x's appearing in this equality; although they are the same number, the x in $\tilde{A}(x)$ is a point in the space X, whereas the x on the right-hand side is a membership value.

Case 1.1: The function $f : X \to Y$ is defined by

$$f(x) = \begin{cases} 4x^2 & \text{if } x \leqslant 0.5, \\ 2(1-x) & \text{if } x > 0.5. \end{cases}$$

It is easy to check that

$$f(\tilde{A})(y) = \begin{cases} 1 & \text{if } y \in A, \\ 1 - y/2 & \text{otherwise ,} \end{cases}$$

where $A = [0,1] \cap \{y \in \mathbb{Q},\ \sqrt{y} \notin \mathbb{Q}\}$. In particular we have $f(\tilde{A})(0.5) = \sup\{0.75, \sqrt{0.125}\} = 1 \notin \{0.75, \sqrt{0.125}\}$. Thus the condition of the corollary is not fulfilled and therefore we conclude that the images of the α-cuts are not the α-cuts of the image. In order to confirm this statement, we can compute directly $f(A_{0.75})$ and $[f(\tilde{A})]_{0.75}$. We obtain

$$f(A_{0.75}) = f([0.75,1] \cap \mathbb{Q}) = [0,0.5] \cap \mathbb{Q},$$

$$f(\tilde{A})_{0.75} = ([0,0.5] \cap \mathbb{Q}) \cup A \neq f(A_{0.75}).$$

Case 1.2: The function $f : X \to Y$ is defined by

$$f(x) = \begin{cases} 2x & \text{if } x \leqslant 0.5, \\ 2(1-x) & \text{if } x > 0.5. \end{cases}$$

It is easy to check that

$$f(\tilde{A})(y) = \sup\left\{\frac{y}{2}, 1 - \frac{y}{2}\right\} = \max\left\{\frac{y}{2}, 1 - \frac{y}{2}\right\}.$$

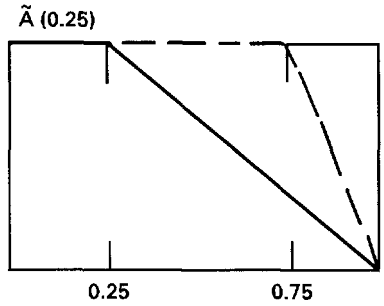

Fig. 1.

The condition of Corollary 1 is fulfilled and therefore the images of the α-cuts coincide with the α-cuts of the image. This fact may be confirmed by means of a direct determination of the two subsets.

Example 2. In this example the sets X, Y are the same as in Example 1, the lattice $(L, \preceq)$ is the family of the maps from $[0,1]$ to $[0,1]$ endowed with the usual ordering between functions (we are dealing with second-order subsets of X and Y); this means that the value of the membership function at a point x is a function: $\tilde{A}(x) = \varphi_x : [0,1] \to [0,1]$. The map f is the same of Example 1, case 2, that is

$$f(x) = \begin{cases} 2x & \text{if } x \leqslant 0.5, \\ 2(1-x) & \text{if } x > 0.5. \end{cases}$$

Case 2.1: The fuzzy set $\tilde{A}$ is given by (see Fig.1)

$$\tilde{A}(x) = \varphi_x(t) = \min\left[1\ ,\ \frac{1}{1-x}(1-t)\right].$$

$$[f(\tilde{A})](y) = \sup\left\{\tilde{A}\left(\frac{y}{2}\right), \tilde{A}\left(1 - \frac{y}{2}\right)\right\} = \tilde{A}\left(1 - \frac{y}{2}\right) = \max\left\{\tilde{A}\left(\frac{y}{2}\right), \tilde{A}\left(1 - \frac{y}{2}\right)\right\},$$

because it is evident that $1 - y/2 \geqslant y/2\ \forall y \in [0,1]$. Moreover $x' < x'' \Rightarrow \tilde{A}(x') \leqslant \tilde{A}(x'')$. We may take as an example the l^*-cut corresponding to second-order fuzzy value $l^*(t) = 1$ if $t \in [0,0.6]$, $l^*(t) = 4 - 5t$ if $t \in [0.6,0.8]$ and $l^*(t) = 0$ if $t \in [0.8,1]$ (Fig. 2). It can be checked, without any difficulty, that $[f(\tilde{A})]_{l^*} = \{y \mid f[\tilde{A}](y) \succeq l^*\} = \{y \mid \tilde{A}(1 - y/2)\} = [0,0.8]$, and $f(A_{l^*}) = \{y = f(x) \mid x \in A_{l^*}\} = \{y = f(x) \mid x \in [0.6,1]\} = [0,0.8]$.

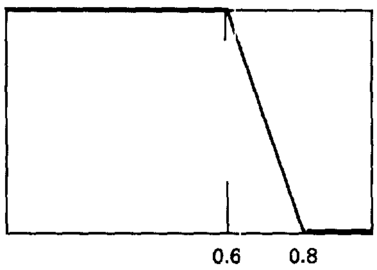

Fig. 2.

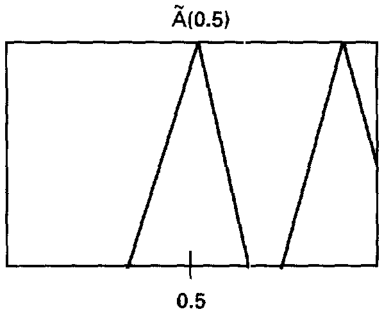

Fig. 3.

Case 2.2: The values $\tilde{A}(x)$ are the functions $\varphi_x(t)$ represented graphically by isosceles triangles of the same shape (eventually cut off at the walls $x = 0$ and $x = 1$), with the base of width $b = 0.2$ centered on point x (see Fig. 3).

It is easy to check that the fuzzy value $[f(\tilde{A})](y) = \sup\{\tilde{A}(y/2), \tilde{A}(1 - y/2)\}$ (see Fig. 4) does not belong to the set $\{\tilde{A}(y/2), \tilde{A}(1 - y/2)\}$ unless $y = 1$. Therefore in general the upper bound is not a maximum and we do not apply Corollary 1 to obtain $[f(\tilde{A})]_l$. As an example let us consider the subsets $f(A_l)$ and $[f(\tilde{A})]_l$ corresponding to the fuzzy value $l \in L$ represented by the isosceles triangle with height $h = 0.2$ centered on point $1/2$ and base width $\beta = 0.3$ (see Fig. 5).

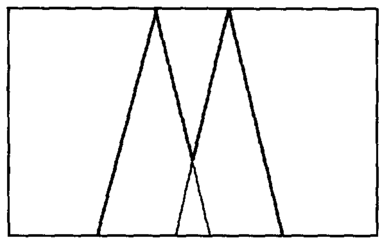

Fig. 4.

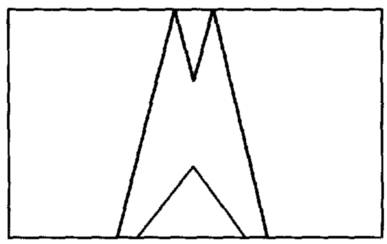

Fig. 5.

It is easy to check that $A_l = \emptyset$ and therefore $f(A_l) = \emptyset$. On the other hand, we can recognize with straightforward computations, that $[f(\tilde{A})]_l$ contains all the crisp values y in the interval $[0.84, 0.90]$ (see Fig. 5).

We can observe that in Example 1 the equality between $[f(\tilde{A})]_l$ and $f(A_l)$ depends on the form of the function f, whereas in Example 2 it depends on the structure of the L-fuzzy subset $\tilde{A}$.

References

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