Asset Pricing with Matrix Jump Diffusions

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Motivation

From Models:

In the early finance literature, prices are assumed to be driven by diffusion processes, mostly geometric Brownian motion.

However, reality differs significantly from what we would expect from such an assumption.

Two main adjustments, resulting in an uncountable amount of different model specifications:

1. Stochastic volatility models (e.g., Heston, 1993).
2. Discontinuous processes (e.g., jump diffusions for returns, Merton, 1976).

...To Frameworks:

Duffie, Pan, and Singleton (2000) provide comprehensive study of affine asset pricing models.

Leippold and Wu (2002) provide a characterization of the quadratic class of asset pricing.

Cheng and Scaillet (2007) extend this class to linear-quadratic jump diffusion models.
What We Observe in the Literature:

- Number of papers are long, identified classes are wide.
- However, real applicable self-coherent examples are few and far between.
- Depending on the specific application, researchers are willing to sacrifice and cut corners to partially account for evidence.

The Reason:

- The affine square root process is not flexible, extensions to multivariate settings inherently difficult.
- Although much more flexible than a multivariate square-root model, the quadratic class does not allow for jumps.
- Adding linear jumps to a quadratic model requires a lot of additional model restrictions.
- Maintaining positive semi-definiteness in the presence of jumps in covariances is a non-trivial exercise.
What We Do

Our Main Contribution

We provide a comprehensive framework to consistently model a **multivariate factor structure** featuring:

- Stochastic volatilities,
- Stochastic correlations,
- Jumps in covariances and returns,

within a **single-asset and multi-asset framework**.

Matrix Affine Jump Diffusion (AJD) Process

- **From vectors to matrices:** We specify a **matrix AJD process** which serves us as our main modeling device.

- **Analytical tractability:** We provide a thorough transform analysis of our model. Laplace transforms are available in (quasi) closed-form.

- **Coherency:** Our coherent framework gives us extra flexibility the stylized facts of financial data and hence opens up a wide range of applications.
Stylized Facts I: Jumps in Equity Markets

From Single Factor Modeling:

- Stochastic volatility as well as jumps in the return process.  
  Andersen et al. (2002), among many others.

- Discontinuities in both price and volatility dynamics. Jump times cluster.  
  Chernov et al. (2003), Eraker et al. (2003), Eraker (2004), among many others.

...To Multi-Factor Structures:

- Single-factor models have shortcomings. Correlation products on the rise!  
  e.g., Carr and Wu (2006), Egloff, Leippold, Wu (2007)

- Ignoring the stochastic component of correlation can lead to erroneous 
  international portfolio allocations. e.g., Ball and Torous (2000)

- Hedging demand under stochastic correlation can be substantially larger than the 
  volatility hedging demand implied by univariate models. e.g., Buraschi, Porchia, Trojani 
  (2007)

▶ Conclusion: We need a multifactor model with flexible covariance structure 
  including jumps, extendable to a multi-asset framework!
Maturities are from 1M to 5Y. Note the impact of the 1998 and 2003 crises. Cross-sectional properties (smile and term structure) may change dramatically through time.

Indication for multiple factors (see Egloff, Leippold, Wu, 2007) and non-diffusive behavior of volatility.
Stylized Facts I: Multiple Sources of Variation

- Implied Volatilities on Single Stock (ABB): Maturities are from 1M to 12M.
- Note the changing sign of the slope at the short end of the volatility surface and the recently observed rebouncing of the volatility wings.

▶ Indication for rather involved factor dependency.
Stylized Facts I: Stochastic Skewness

Risk Reversals (RR)
- Definition:
  \[ RR = IV(C_{10\%}^{otm}) - IV(P_{10\%}^{otm}). \]
- Captures the slope of the smile.
- Slope proxies the skewness of the risk-neutral return distribution.
- Illustration using options with 1M (red) and 12M (blue) maturities.
- Skewness is highly stochastic, indicating a stochastic correlation between return and volatilities.
- Equity skewness may change sign!
- Remark: Equity skewness is highly correlated to CDS spreads.
Stylized Facts II: Jumps in Term Structure

- Interest rates jump.
  e.g. Ball and Torous (1985)

- Unspanned volatility phenomena.
  e.g. Collin-Dufresne and Goldstein (2002)

- Important source of USV may be the presence of jump risk.
  Wright and Zhou (2008)

- Bipower variation test on GovPX data: Strong evidence of jumps with stochastic intensity. May be crucial to break down CLT to account for caps/floors data.

▶ Conclusion: We need a multifactor model with flexible covariance structure including jumps with stochastic intensity!
The Matrix Process

Basic Setup

- The adapted Markov process $X \in S_n^+$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ solves:

$$dX_t = (\Omega \Omega' + MX_t + X_t M') \, dt + \sqrt{X_t} \, dB_t \, Q + Q' \, dB'_t \sqrt{X_t} + dJ_t.$$ 

- $B$ is a Brownian motion in $\mathbb{R}^{n \times n}$ and $\Omega, M, Q \in \mathbb{R}^{n \times n}$.

- $J$ is pure jump process with IID jump sizes $\xi X \sim \nu X \in S_n^+$ and affine intensity $\lambda X(x) = \lambda X,0 + tr(\lambda X,1x)$, $\lambda X,0 \geq 0$, $\lambda X,1 \in S_n^+$.

Remarks

- For $\lambda X(x) \equiv 0$, we obtain the Wishart process introduced by Bru (1991) and studied in Gouriéroux and Sufana (2004).

- If $\Omega \Omega' = Q = 0_{n \times n}$, $X$ is a pure jump process in the class of Ornstein-Uhlenbeck matrix subordinators analyzed by Barndorff-Nielsen and Stelzer (2007).

- There are many candidate distributions for the jump sizes of $J \in S_n^+$. 
Implied Correlation Process

- Define the correlation as $\rho_{ij} := \frac{X_{ij}}{\sqrt{X_{ii}X_{jj}}}$, $1 \leq i \leq j \leq n$. Then,
\[
d\rho_{ij} = \left( A_{ijt}\rho_{ijt}^2 + B_{ijt}\rho_{ijt} + C_{ijt} \right) dt + \frac{e'_i \sqrt{X_t} dB_t Q_j + e'_j \sqrt{X_t} dB_t Q_i}{\sqrt{X_{ii}X_{jj}}} - \rho_{ijt} \left( \frac{e'_i \sqrt{X_t} dB_t Q_i}{X_{ii}} + \frac{e'_j \sqrt{X_t} dB_t Q_j}{X_{jj}} \right) + \rho_{ijt} \zeta_{ij}^X dJ^\rho_t,
\]
where $J^\rho$ is a pure jump process.
- The drift is quadratic in $\rho_{ijt}$ with stochastic coefficients depending only on $X_{kk}$ and $\rho_{kl}$, $1 \leq k, l \leq n$.
- The correlation has non-linear with positive and negative jumps with size
\[
\zeta_{ij}^X = \frac{1 + \frac{\xi_{ij}}{X_{ij}}}{\sqrt{\left(1 + \frac{\xi_{ii}}{X_{ii}}\right)\left(1 + \frac{\xi_{jj}}{X_{jj}}\right)}} - 1.
\]
Implied Correlation Process (Con’d)

Volatility Dynamics

Correlation Dynamics

Stochastic Intensity

Correlation Jumps
Discounted Laplace Transforms

- Let $R(x) = \rho_0 + \text{tr}(\rho_1 x)$, $\rho_0 \geq 0$, $\rho_1 \in S_n^+$. Then, $R(x) \geq 0$.
- We can define the discounted Laplace transform as:
  \[
  \psi^X(\Gamma, X_t, t, T) = \mathbb{E} \left[ e^{-\int_t^T R(X_s)ds + \text{tr}(\Gamma X_T)} | \mathcal{F}_t \right] = e^{B(T-t) + \text{tr}(A(T-t)X_t)},
  \]
  with $\Gamma \in S_n$.
- The coefficients $B(u) \in \mathbb{R}$, $A(u) \in S_n^+$ solve the matrix Riccati equations (with $\Theta^X$ the Laplace transform of $\xi^X$):
  \[
  \frac{dA(\tau)}{d\tau} = -\rho_1 + M' A(\tau) + A(\tau) M + 2A(\tau)Q'QA(\tau) + \lambda_{X,1}[\Theta^X(A(\tau)) - 1],
  \]
  \[
  \frac{dB(\tau)}{d\tau} = -\rho_0 + \text{tr}(A(\tau)\Omega\Omega') + \lambda_{X,0}[\Theta^X(A(\tau)) - 1],
  \]
  subject to $B(0) = 0$ and $A(0) = \Gamma$. 
Given the Laplace transform, we can follow Duffie, Pan, and Singleton (2000) to price options within our framework.

Define

\[ G_{A,B}(y; X_0, T) := \mathbb{E} \left[ \exp \left( - \int_0^T R(X_s) ds + \text{tr}(AX_T) \right) \mathbb{I}_{\text{tr}(BX_t) \leq y} \right], \]

where \( \mathbb{I}_C \) is the indicator function of event \( C \), \( y \in \mathbb{R} \), and \( A, B \in S_n \).

The Fourier-Stieltjes transform of \( G_{A,B} \), if well defined, is given by:

\[ G_{A,B}(v; X_0, T) = \int_{\mathbb{R}} \exp(iv) dG_{A,B}(y; X_0, T) \]

\[ = \mathbb{E} \left[ \exp \left( - \int_0^T R(X_s) ds + \text{tr}((A + ivB)X_t) \right) \right] \]

\[ = \Psi^X(A + ivB, X_0, 0, T), \]

where \( v \in \mathbb{R} \) and \( i = \sqrt{-1} \).
Transform Inversion Formula (Con’d)

- Under regularity conditions, the transform $G_{A,B}$ is:

$$G_{A,B}(y; X_0, T) = \frac{\psi^X(A, X_0, 0, T)}{2} - \frac{1}{\pi} \int_0^\infty \text{Im} \left[ \frac{\psi^X(A + ivB, X_0, 0, T)e^{-ivy}}{v} \right] dv,$$

where $\text{Im}(c)$ is the imaginary part of $c \in \mathbb{C}$.

**Remark**

- We thus extended the range of known transform solutions in Duffie, Pan, and Singleton (2000) to general matrix AJD processes.
- For $R(X_t) = 0$ and $A = 0$, we can now calculate the conditional probability distribution of $tr(BX_T)$ given $X_0$.
- The corresponding density of $tr(BX_T)$ follows by differentiation of $G_{A,B}(\cdot; X_0, T)$. 
Risk Neutral Pricing

- We define an exponentially affine stochastic discount factor as

\[ \xi_t = \exp \left( - \int_0^t R(X_s) ds \right) \exp(\alpha(t, T) + \text{tr}(\beta(t, T)X_t)), \]

for continuous functions \( \alpha(\cdot, T) : [0, T] \to \mathbb{R} \), \( \alpha(0, T) = 0 \), and \( \beta(\cdot, T) : [0, T] \to S_n^+, \beta(0, T) = 0 \).

- Under the risk-neutral probability \( \mathbb{P}^* \) AJD process is:

\[
dX_t = (\Omega \Omega' + M^*(t)X_t + X_tM^*(t))dt + \sqrt{X_t}dB_t^* + QdJ_t
\]

where \( J \in S_n^+ \) with jump sizes \( \xi^X_t \overset{d}{=} \nu^X(t) \) and affine intensity

\[ \lambda^*_X(t, X_t) = \lambda^*_{X,0}(t) + \text{tr}(\lambda^*_{X,1}(t)X_t). \]

- The additional parameters are:

  - \( M^*(t) = M + 2Q'Q\beta(t, T) \),
  - \( \lambda^*_{X,0}(t) = \lambda^*_{X,0}\Theta^X(\beta(t, T)) \),
  - \( \lambda^*_{X,1}(t) = \lambda^*_{X,1}\Theta^X(\beta(t, T)) \),
  - \( \Theta^X(\Gamma, t) = \Theta^X(\Gamma + \beta(t, T))/\Theta^X(\beta(t, T)) \).
Let \((L, J)\) be a pure jump process with IID jump sizes \((\xi^Y, \xi^X)\) in \(\mathbb{R} \times S_n^+\) and Laplace transform \(\Theta^{YX}\).

Jump sizes follow finite joint probability distribution \(\nu^{YX} = \nu^Y|X \nu^X\) on \(\mathbb{R} \times S_n^+\) and affine intensity

\[
\lambda_{YX}(X_t) = \lambda_{YX,0} + \text{tr}(\lambda_{YX,1}X_t), \quad \lambda_{YX,0} \geq 0, \lambda_{YX,1} \in S_n^+.
\]

The dynamics for the single return process \(Y_t\) are:

\[
dY_t = \left[ R(X_t) + \mu_e(X_t) - \frac{1}{2}\text{tr}(X_t) \right] dt + \text{tr}\left( \sqrt{X_t} dZ_t \right) + dL_t.
\]

Here, \(\mu_e(X_t) = \mu_{e,0} + \text{tr}(\mu_{e,1}X_t), \mu_{e,0} \in \mathbb{R} \) and \(\mu_{e,1} \in S_n\), is an affine function of \(X_t\).

To introduce the leverage effect, we define the Brownian motion \(Z \in \mathbb{R}^{n \times n}\) as

\[
Z_t = B_t P' + W_t \sqrt{I_n - PP'}, \quad P \in \mathbb{R}^{n \times n},
\]

where \(W \in \mathbb{R}^{n \times n}\) is another standard Brownian motion, independent of \(B\).
Define $V_t := \text{Var}_t(dY_t)/dt$ and $H := I_n + \lambda YX,1 \mathbb{E}[(\xi Y)^2]$. Then:

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<th>$\lambda YX,0 = 0, \lambda YX,1 = 0$</th>
<th>Unconstrained</th>
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</thead>
<tbody>
<tr>
<td>$\text{Var}_t(dY_t)/dt$</td>
<td>$\text{tr}(X_t)$</td>
<td>$\text{tr}(X_t) + \lambda YX(X_t)\mathbb{E}[(\xi Y)^2]$</td>
</tr>
<tr>
<td>$\text{Var}_t(dV_t)/dt$</td>
<td>$4\text{tr}(Q'QX_t)$</td>
<td>$4\text{tr}(HQ'QHX_t) + \lambda YX(X_t)\mathbb{E}[\text{tr}(H\xi X)^2]$</td>
</tr>
<tr>
<td>$\text{Cov}_t(dY_t, dV_t)/dt$</td>
<td>$2\text{tr}(PQX_t)$</td>
<td>$2\text{tr}(PQHX_t) + \lambda YX(X_t)\mathbb{E}[\xi Y \text{tr}(H\xi X)]$</td>
</tr>
</tbody>
</table>

Hence, the correlation in the no-jump case becomes

$$\text{Corr}_t(dY_t, dV_t) = \text{tr}(PQX_t)/\sqrt{\text{tr}(X_t)\text{tr}(Q'QX_t)},$$

which already provides a rich structure for the leverage dynamics and generalizes Christoffersen, Heston, and Jacobs (2007).

However, such a model specification does not account for the evidence of jumps in returns and volatilities.
Laplace Transform

• The discounted Laplace transform of $Y_T$ has the exponentially affine form:

$$
\Psi^Y(\gamma) := \mathbb{E} \left[ \exp(- \int_t^T R(X_s) ds + \gamma Y_T) \right]
= \exp(\gamma Y_t) \exp(B(t - T) + tr(A(T - t)X_t)).
$$

• Functions $B(u) \in \mathbb{R}$ and $A(u) \in S_n^+$ solve the system of matrix differential Riccati equations:

$$
\frac{\partial A}{\partial \tau} = (\gamma - 1)\rho_1 + \gamma \mu_{e,1} + \frac{\gamma(\gamma - 1)}{2} I_n + A(\tau)(M + \gamma Q'P') + (M' + \gamma PQ)A(\tau) + 2A(\tau)Q'QA(\tau) + \lambda YX,1[\Theta^{YX}(\gamma, A(\tau)) - 1]
$$

$$
\frac{\partial B}{\partial \tau} = (\gamma - 1)\rho_0 + \gamma \mu_{e,0} + tr(\Omega\Omega'A(\tau)) + \lambda YX,0[\Theta^{YX}(\gamma, A(\tau)) - 1]
$$

subject to terminal conditions $B(0) = 0$ and $A(0) = 0$. 

Asset Pricing with Matrix Jump Diffusions
Application I: Multifactor Option Pricing

Calibrated Call Prices

Model
MAJD(2), diagonal $\xi^X$, constant intensity.

Method
2 Step procedure, starting with a genetic algorithm and nested optimization.

Data
174 S&P Index Options on 2001/1/12. $S_0 = 1,483$. Maturities from 1W to 1Y. Moneyness $\pm 50\%$. 
Application I: Multifactor Option Pricing (Con’d)

Parameter Values:

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</thead>
<tbody>
<tr>
<td>$M$</td>
<td>$Q$</td>
<td>$P$</td>
<td>$\lambda_0$</td>
<td>$\xi^X$</td>
<td>$k$</td>
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<td>-4.1540</td>
<td>0</td>
<td>0.2537</td>
<td>0.0048</td>
<td>-0.8430</td>
<td>-0.1126</td>
<td>0.11</td>
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<tr>
<td>-0.3472</td>
<td>-1.7851</td>
<td>0</td>
<td>0.2417</td>
<td>0</td>
<td>-0.6300</td>
<td>0</td>
</tr>
</tbody>
</table>

Mean abs $\$ error ... 0.471
Within bid-ask .... 84%
CCMY error ....... 0.015
Application I: Multifactor Option Pricing (Con’d)

Implied Volatility Smiles and Skewness

- Adding correlation between volatility factors extends the span of possible smiles and skewness patterns.
- With the jump component, we get additional mileage for out-of-the-money puts.
Application I: Multifactor Option Pricing (Con’d)

Implied Volatility Smiles and Skewness

- 30-days smiles for S&P500 options from Aug-1994 to May-2005, for which $IV(K/S = 0.975) \in [0.187, 0.189]$.
- 30-days 10% risk reversals such that $IV(ATM) \in [0.1625, 0.1725]$. 
Application I: Multifactor Option Pricing (Con’d)

Implied Volatility Term Structure Shapes

- Implied volatility term structures within the model and in the data, stratified across three volatility regimes, in the period between August 2, 1994 and May 31, 2005.
Double-Jump Matrix AJD Process for Multiple Assets

- Let \((L, J)\) be a pure jump process with values with IID jump sizes \((\xi^Y, \xi^X)\) in \(\mathbb{R}^n \times S^+_n\) and Laplace transform \(\Theta^{YX}(\gamma, \Gamma)\).

- Jump sizes follow a finite joint probability distribution \(\nu^{YX} = \nu^Y|_X \nu^X\) on \(\mathbb{R}^n \times S^+_n\) with affine intensity
  \[
  \lambda_{YX} = \lambda_{YX,0} + \text{tr}(\lambda_{YX,1}X_t), \quad \lambda_{YX,0} \geq 0, \quad \lambda_{YX,1} \in S^+_n.
  \]

- \(\mathbb{P}\)-dynamics of return \(Y_t \in \mathbb{R}^n\) are:
  \[
  dY_t = \left[ R(X_t)1_n + \mu_e(X_t)\beta + X_t\eta - \frac{1}{2} \text{diag}[X_t] \right] dt + \sqrt{X_t}dZ_t + dL_t.
  \]
  where \(\mu_e(X_t) = \mu_{e,0} + \text{tr}(\mu_{e,1}X_t)\), with \(\mu_{e,0} \in \mathbb{R}\) and \(\mu_{e,1} \in S_n\), \(\beta, \eta \in \mathbb{R}^n\), and \(\text{diag}[X] \in \mathbb{R}^n\) with \(i\)-th component equal to \(X_{ii}\).

- To introduce leverage, we define the Brownian motion \(Z \in \mathbb{R}^n\) as:
  \[
  Z_t = B_t\rho + \sqrt{1 - \rho'\rho}W_t, \quad \rho'\rho \leq 1,
  \]
Leverage in the Multi-Asset Case

- Define $H_{ij} := e_j e_i' + \lambda YX_{1} E\left[\xi_i \xi_j'\right].$
- The correlation in the no-jump case becomes a constant:
  $$\text{Corr}_t(dY_{it}, dV_{iit}) = \rho' Q_i \sqrt{Q_i' Q_i}. $$
- Only jumps generate stochastic leverage structure while preserving a parsimonious affine structure:
  $$\leadsto$$ Crucial for pricing multi-asset options and correlation products!

<table>
<thead>
<tr>
<th>$\text{Cov}<em>t(dY</em>{it}dY_{jt})$</th>
<th>$\lambda YX_{0} = 0, \lambda YX_{1} = 0$</th>
<th>Unconstrained</th>
</tr>
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<tbody>
<tr>
<td>$\frac{\text{Cov}<em>t(dY</em>{it}dY_{jt})}{dt}$</td>
<td>$X_{ijt}$</td>
<td>$X_{ijt} + \lambda YX(X_t) E\left[\xi_i \xi_j'\right]$</td>
</tr>
<tr>
<td>$\text{Var}<em>t(dV</em>{iit}dV_{jjt})$</td>
<td>$4X_{ijt} Q_i' Q_j$</td>
<td>$4\text{tr}(H_{ij} Q_i' QH_{ij} X_t) + \lambda YX(X_t) E\left[\text{tr}(H_{ij} X^2)^2\right]$</td>
</tr>
<tr>
<td>$\text{Cov}<em>t(dY</em>{it}dV_{iit})$</td>
<td>$\rho'(X_{iit} Q_j + X_{ijt} Q_j)$</td>
<td>$\rho' Q(H_{ij} + H_{ij}') X_t e_i + \lambda YX(X_t) E\left[\xi_i' \text{tr}(H_{ij} X^2)\right]$</td>
</tr>
</tbody>
</table>
Application II: Portfolio Allocation

Basic Setup

- Jumps in returns play an economically significant role, overstating diversification effects. 
  
  Das and Uppal (2004)

- To isolate the impact of jumps in the covariance process on portfolio allocation, we assume no jumps in the associated price processes.

- We consider a CRRA utility investor optimizing wealth $W_T$ by allocating in $n$ risky assets with $\mathbb{P}$-dynamics:

$$dS_t = \text{diag} [S_t] \left( [R(X_t)1_n + X_t \eta] dt + \sqrt{X_t} dZ_t \right),$$

where $X$ follows a matrix AJD process and $X_t \eta$ represents the excess return vector.

- $Z$ is a $n \times 1$ standard Brownian motion defined by:

$$Z_t = B_t \rho + \sqrt{1 - \rho' \rho} W_t, \quad \rho' \rho \leq 1.$$

- For simplicity, we set $R(X_t) = r$. 

Application II: Portfolio Allocation (Con’d)

Optimal Portfolios

- The optimal portfolio weights are
  \[ w^* = \frac{1}{\gamma} \left( \eta + 2A(\tau)Q'\rho \right), \]

- Coefficients \( A(\tau) \) and \( B(\tau) \) solve, subject to \( A(0) = B(0) = 0 \), the ordinary differential equations:
  \[
  A'(\tau) = A(\tau) \left( M + \frac{1 - \gamma}{\gamma} Q'\rho\eta' \right) + \left( M' + \frac{1 - \gamma}{\gamma} \eta\rho' Q \right) A(\tau) \\
  + 2A(\tau)Q' \left( I + \frac{1 - \gamma}{\gamma} \rho\rho' \right) QA(\tau) - \frac{1 - \gamma}{2\gamma} \eta\eta' \\
  + \lambda_{X,1}[\Theta^X(A(\tau)) - 1], \\
  B'(\tau) = (1 - \gamma)r + tr \left[ \Omega\Omega^\top A(\tau) \right] + \lambda_{X,0}[\Theta^X(A(\tau)) - 1].
  \]

\[ \Rightarrow \text{Only jumps with stochastic intensity matter!} \]
Application II: Portfolio Allocation (Con’d)

Parameter Choice for Jump Risk


- Adding a little bit of jump:

  ![Graphs showing Return Volatility Difference and Correlation Difference over time](image)

- Covariance matrix exhibits a mean jump intensity of 1.9%.
- Note the time-varying intensity and the small jump size.
Hedging demand for a reasonable amount of jump risk (solid lines) and for no jump risk (dashed lines).

Total hedging demand is expressed in percentage of the myopic portfolio.
Model Design

- An affine model of the term structure can be defined by specifying the short rate as

\[ R(t) = \rho_0 + \text{tr}[\rho_1 X_t], \]

where \( X \) follows a matrix AJD under \( \mathbb{P}^* \).

- Then bond prices are just given by \( \Psi^X(0, X_t, t, T) \) and yields are affine in \( X_t \).

- Bond yields are stochastically correlated and subject to common jumps, for which the imperfectly correlated jump sizes are generated by an IID \( n(n + 1)/2 \)
  dimensional random matrix \( \xi^X \).

- Yields covariances \( \text{Cov}_t (dR(t, T_1), dR(t, T_2)) \) are given by:

\[
\text{tr}[A(\tau_1)Q'QA(\tau_2)X_t] + \mathbb{E}_t [\text{tr}(A(\tau_1)dJ_t)\text{tr}(A(\tau_2)dJ_t)]
\]

\[
\frac{\tau_1 \tau_2}{\tau_1 \tau_2},
\]

with \( \tau_k = T_k - t \).
Application III: Term Structure Modeling (Con’d)

Simulation

- We simulate a 3-factor model (parameters in the appendix).
- Interest rate vol is at 2%.
- We choose affine jump intensity.
- Jump mean: 0.034%; Jump volatility: 0.053%; Mean intensity: 0.08.
- Interest rate process exhibits jumps and a stochastic covariance structure between factors.
We introduce an analytical yet flexible new class of multivariate affine models to account for the most prominent features of financial time series within a multivariate setting.

When we account for jumps in the covariance process and in the return processes, we can retain all desirable features of the model also in a multi-asset setting (crucial for pricing correlation products, quantos, basket options, etc.).

Given the flexibility and universality of our model setup, our model might open up a wide range of interesting applications.

In the future, we will explore our model framework specifically along different lines, including multi-asset option pricing, term structure modeling, and multivariate portfolio choice.