Optimal Consumption Policies in Illiquid Markets

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joint work with:

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Outline

1. Pham-Tankov model and problem formulation.
2. Dynamic Programming and first-order coupled nonlinear IPDE.
3. Regularity results: C^1 regularity for the value functions.
4. Existence and characterization of optimal strategies:
    - feedback representation form;
    - solution of the Euler-Lagrange ODE.
5. Numerical illustrations.

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1. The Pham-Tankov model and problem formulation

1.a The Pham-Tankov model of liquidity risk

Optimal portfolio/consumption problem:

- *The investor has access to a market in which an illiquid asset (stock or fund) is traded*;
- *The asset price is observed only at random times*;
- *Discrete trading is possible only at these random times*;
- *The investor may consume continuously from her/his cash holding.*

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Nonstandard mixed discrete/continuous stochastic control problem

$\Longrightarrow$ new type of dynamic programming equation.

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The model is first introduced in the paper:

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where they investigated various theoretical issues, including the viscosity characterization of the value functions.

Then, a convergent numerical algorithm to compute the value functions and several numerical illustrations are provided in the paper:

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1.b Problem formulation

- Stock price S is observed and traded only at exogenous random times $(\tau_k)_{k\geq 0}$ with $\tau_0 = 0 < \tau_1 < \ldots < \tau_k < \ldots$;
- The investor may consume continuously from the bank account between two trading dates.

Continuous observation filtration

$$\mathbb{G}^c = (\mathcal{G}_t)_{t\geq 0}, \quad \mathcal{G}_t = \sigma\{(\tau_k, S_{\tau_k}) \ : \tau_k \leq t\}$$

Discrete observation filtration

$$\mathbb{G}^d = (\mathcal{G}_{\tau_k})_{k\geq 0}.$$

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- **Control policy** : mixed discrete/continuous time process (α, c) :
  - $\alpha = (\alpha_k)_{k \geq 1}$ is a real-valued $\mathbb{G}^d$-predictable process :

    α_k represents the *amount of stock* invested for the period $(\tau_{k-1}, \tau_k]$ after observing the stock price $S_{\tau_{k-1}}$ at time τ_{k-1}
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- **Wealth process** : starting from an initial capital $x \geq 0$, and given a strategy (α, c), the wealth X_k^x of the investor at time τ_k is :

- **Admissible control policy** : given $x \geq 0$, we say that (α, c) is admissible, and we denote $(\alpha, c) \in \mathcal{A}(x)$, if :

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- **Assumption (1).** Conditions on (τ_k, Z_k) :
  - a) $\{\tau_k\}_{k\geq 1}$ is the sequence of jump times of a Poisson process with intensity λ.
  - b) (i) For all $k \geq 1$, conditionally on the interarrival time $\tau_k - \tau_{k-1} = t$, Z_k is independent from $\{\tau_i, Z_i\}_{i<k}$ and has a distribution $p(t, \mathrm{d}z)$.

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    Go
  - c) $\int zp(t, \mathrm{d}z) \geq 0$, $\forall t \geq 0$ and there exist some $\kappa, b \in \mathbb{R}_+$ s.t.

    $$\int (1 + z)p(t, \mathrm{d}z) \leq \kappa e^{bt}, \quad \forall t \geq 0.$$
  - d) The following continuity condition is fulfilled by the measure $p(t, \mathrm{d}z)$:

    $$\lim_{t \to t_0} \int w(z)p(t, \mathrm{d}z) = \int w(z)p(t_0, \mathrm{d}z), \quad \forall t_0 \geq 0,$$

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- **Typical Examples**:

  - ★ S extracted from a Black-Scholes model : $\mathrm{d}S_t = bS_t\mathrm{d}t + \sigma S_t \mathrm{d}W_t$. Then $p(t, \mathrm{d}z)$ is the distribution of

    $$Z(t) = \exp\left[\left(b - \frac{\sigma^2}{2}\right)t + \sigma W_t\right] - 1,$$

    with support $(-1, \infty)$, and condition c) of Assumption (1) is clearly satisfied : $\int (1+z)p(t, \mathrm{d}z) = e^{bt}$.

  - ★ Return process Z_k independent of waiting times $\tau_k - \tau_{k-1}$, so $p(t, \mathrm{d}z) \equiv p(\mathrm{d}z)$. In particular $p(\mathrm{d}z)$ may be a discrete distribution with finite support in $(-1, +\infty)$.

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- **Optimal portfolio/consumption problem**:

Value function

$$v(x) = \sup_{(\alpha,c)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t} U(c_t)\mathrm{d}t\right], \qquad x \geq 0.$$

Assumption (2)

- Utility function $U:\mathbb{R}_+\to\mathbb{R}$, $U(0)=0$, C^1, strictly increasing, strictly concave, satisfying the Inada conditions $U'(0)=\infty$, $U'(\infty)=0$, and the growth condition

$$U(w) \leq K_1 w^{\gamma}, \quad \gamma \in (0,1).$$

- Discount factor ρ such that

$$\rho > b\gamma + \lambda\left(\frac{\kappa^{\gamma}}{\underline{z}^{\gamma}} - 1\right).$$

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2. Dynamic Programming

- Dynamic programming is a mathematical technique for making a sequence of interrelated decisions, which can be applied to many optimization problems (including optimal control problems). Ex: optimal investment problems, consumer maximization problems in macroeconomic models, etc.

- Basic idea: consider a family of optimal control problems with different initial times and states and establish relationships among these problems via the so-called Hamilton-Jacobi-Bellman equation (HJB), which is a nonlinear first-order (in the deterministic case) or second-order (in the stochastic case) partial differential equation.
  If the HJB equation is solvable, then one can obtain an optimal feedback control by taking the maximizer/minimizer of the Hamiltonian or generalized Hamiltonian involved in the HJB equation. This is the so-called *verification technique*.

- It is required that the HJB equation admits *classical solutions*, meaning that the solutions have to be smooth enough.

- To overcome this difficulty a weaker definition of solutions was introduced. *Viscosity solutions*: a kind of nonsmooth solutions to partial differential equations, where the definition of solution is given in terms of sub-superdifferential of the function.
  This concept of solution usually provides a characterization of the value function of the control problem as unique solution of the HJB equation.

- Weak point: such a characterization is not good to use in order to construct optimal controls and regularity results for this kind of solutions are needed to go ahead.

2.a Dynamic Programming Principle (DPP)

Relation on the value function by considering two consecutive trading dates :

$$\begin{aligned} v(x) &= \sup_{(\alpha,c)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t)\mathrm{d}t + e^{-\rho\tau_1} v(X_1^x)\right], \\ &= \sup_{(a,c)\in A_d(x)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t)\mathrm{d}t + e^{-\rho\tau_1} v\left(x - \int_0^{\tau_1} c_t \mathrm{d}t + aZ_1\right)\right], \end{aligned}$$

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$A_d(x)$: pair of deterministic constants a and nonnegative deterministic processes $c = (c_t)_{t\geq 0}$ s.t. : $x - \int_0^{\tau_1} c_t\mathrm{d}t + aZ_1 \geq 0$ a.s. , i.e.

$$-\frac{x}{\bar{z}} \leq a \leq \frac{x}{\underline{z}}$$

$$x - \int_0^t c_u\mathrm{d}u \geq l(a), \quad \forall t\geq 0, \text{ with } \quad l(a) = \max(a\underline{z}, -a\bar{z}) \quad (c\in\mathcal{C}_a(x)).$$

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$$x - \int_0^t c_u \mathrm{d}u \geq -az, \quad \forall t \geq 0, \quad \forall z \in (-\underline{z}, \bar{z}) \quad (c \in \mathcal{C}_a(x)).$$

Under conditions a) and b) of Assumption (1) on (τ_1, Z_1) Ass (1), we compute the expectation :

$$v(x) = \sup_{\substack{a \in \left[-\frac{x}{\bar{z}}, \frac{x}{\underline{z}}\right] \\ c \in \mathcal{C}_a(x)}} \int_0^{+\infty} e^{-(\rho+\lambda)s} \left[U(c_s) + \lambda \int v\left(x - \int_0^s c_u \mathrm{d}u + az \right) p(s, \mathrm{d}z) \right] \mathrm{d}s.$$

- **Dynamic auxiliary deterministic control problem on consumption :**

  $$\hat{v}(t, x, a) = \sup_{c \in \mathcal{C}_a(t,x)} \int_t^{+\infty} e^{-(\rho+\lambda)(s-t)} \left[U(c_s) + \lambda \int v(Y_s^{t,x} + az) p(s, \mathrm{d}z) \right] \mathrm{d}s,$$

  where $\mathcal{C}_a(t, x)$ is the set of deterministic nonnegative processes $c = (c_s)_{s \geq t}$, s.t.

  $$Y_s^{t,x} = x - \int_t^s c_u \mathrm{d}u \geq l(a), \quad s \geq t,$$

  for $(t, x, a) \in \mathcal{D} := \mathbb{R}_+ \times \mathcal{X}$, with $\mathcal{X} = \{(x, a) \in \mathbb{R}_+ \times A : \; x \geq l(a)\}$, by setting $A = \mathbb{R}$ if $\bar{z} < +\infty$ and $A = \mathbb{R}_+$ if $\bar{z} = +\infty$.

Under conditions a) and b) of Assumption (1) on (τ_1, Z_1) Ass (1), we compute the expectation :

$$v(x) = \sup_{\substack{a \in \left[-\frac{x}{\bar{z}}, \frac{x}{\underline{z}}\right] \\ c \in \mathcal{C}_a(x)}} \int_0^{+\infty} e^{-(\rho+\lambda)s} \left[U(c_s) + \lambda \int v\left(x - \int_0^s c_u \mathrm{d}u + az\right) p(s, \mathrm{d}z) \right] \mathrm{d}s.$$

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2.b The equivalent *coupled* deterministic optimization problem

- For each $a \in A$, a deterministic control problem :

$$\hat{v}(t,x,a) = \sup_{c \in \mathcal{C}_a(t,x)} \int_t^{+\infty} e^{-(\rho+\lambda)(s-t)} \left[U(c_s) + \lambda \int v(Y_s^{t,x} + az) p(s, \mathrm{d}z) \right] \mathrm{d}s,$$

- A classical (scalar) maximum problem on a concave function

$$v(x) = \sup_{a \in \left[-\frac{x}{\bar{z}}, \frac{x}{\underline{z}}\right]} \hat{v}(0,x,a), \quad x \geq 0,$$

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2.c The system of coupled nonlinear IPDE

- The Hamilton-Jacobi (HJ) equation associated to the first problem is:

$$(\rho+\lambda)\hat{v} - \frac{\partial \hat{v}}{\partial t} - \tilde{U}\left(\frac{\partial \hat{v}}{\partial x}\right) - \lambda \int v(x+az)p(t,\mathrm{d}z) = 0, \quad (t,x,a) \in \mathcal{D}, \quad (1)$$

where $\tilde{U}(p) = \sup_{c>0}[U(c) - cp],\ p \geq 0$ is the convex conjugate of U.

- This is coupled with the above:

$$v(x) := \sup_{a \in \left[-\frac{x}{z},\, \frac{x}{z}\right]} \hat{v}(0,x,a), \quad x \geq 0, \qquad (2)$$

Dynamic programming ⟶ system of coupled nonlinear IPDE (1)-(2).

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2.d Some properties of $v, \hat{v}$

Under Assumptions (1)-(2), the following properties hold:

- v is nondecreasing, concave, continuous on $\mathbb{R}_+$ with $v(0) = 0$;
- $\hat{v}$ is concave in (x, a) and continuous on $\mathcal{D}$;
- **Growth condition** (G1) : there exists some positive constant K s.t.

$$\hat{v}(t, x, a) \leq K(e^{bt}x)^{\gamma}, \quad \forall (t, x, a) \in \mathcal{D},$$
$$v(x) \leq Kx^{\gamma}, \quad \forall x \geq 0.$$

- **Boundary data** (B1) $\longleftrightarrow$ nonnegative wealth constraint :

$$\hat{v}(t, x, a) = \lambda \int_t^{+\infty} e^{-(\rho+\lambda)(s-t)} \int v(x+az)p(s, \mathrm{d}z)\mathrm{d}s, \quad \forall t \geq 0, \ (x, a) \in \partial\mathcal{X}$$

◂ go

- v is strictly increasing on $\mathbb{R}_+$;
- $\hat{v}$ is strictly increasing in $x \geq l(a)$, given $a \in A$;
- **The scaling relation for power utility**: in the case where

$$U(x) = K_1 x^{\gamma}, \quad 0 < \gamma < 1,$$

$$\hat{v}(t, \beta x, \beta a) = \beta^{\gamma}\hat{v}(t, x, a), \qquad v(\beta x) = \beta^{\gamma} v(x).$$

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3. Regularity results

- In the paper [PT 07(a)] the viscosity solution approach is developed giving existence and uniqueness, but no regularity result is proved, so the *verification technique* cannot be used to obtain an optimal control in feedback form.

- In the present paper we prove that the viscosity solution is indeed *regular* in a wide class of cases that includes the examples.

- This allows to get the existence of the optimal control that we characterize both in feedback form in terms of the derivatives of the value functions and as the solution of a second-order ODE.

3.a Stationary case

Assume that the distribution $p(t, \mathrm{d}z)$ is *time independent*.

$$\boxed{p(t, \mathrm{d}z) \equiv p(\mathrm{d}z).}$$

Then the coupled HJ becomes:

$$(\rho + \lambda)\hat{v} - \tilde{U}\left(\frac{\partial \hat{v}}{\partial x}\right) - \lambda \int v(x + az)p(\mathrm{d}z) = 0, \quad t \geq 0, x \geq l(a), \tag{3}$$

$$v(x) = \sup_{a \in \left[-\frac{x}{z}, \frac{x}{z}\right]} \hat{v}(x, a), \quad x \in \mathbb{R}_+, \tag{4}$$

Theorem

Suppose Assumptions (1)-(2) are satisfied. Then

- ★ $\forall a \geq 0$ *we have* $\hat{v}(\cdot, a) \in C^2(l(a), +\infty)$ *and* $\frac{\partial \hat{v}}{\partial x}(l(a)^+, a) = +\infty$.
- ★ *We have* $v \in C^1(0, +\infty)$ *and* $v'(0^+) = +\infty$.

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▶ Idea of the proof

First we prove that $\hat{v}(\cdot, a) \in C^1(l(a), +\infty)$. The other regularity properties follow by a kind of bootstrap argument based on the coupling.

Arguing by contradiction, we use:

- the *viscosity characterization* of the value functions to the original control problem by means of viscosity solution to the coupled IPDE (see [PT 07(a)]).
- the concavity of the function $\hat{v}(\cdot, a)$ together with the strict convexity of the convex conjugate $\tilde{U}$ with similar arguments to those given in:

  [BCD 97] Bardi M. and Capuzzo-Dolcetta I., OPTIMAL CONTROL AND VISCOSITY SOLUTIONS OF HAMILTON-JACOBI-BELLMAN EQUATIONS. Birkhäuser Boston Inc., Boston, MA.

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3.b Nonstationary case

To use the same argument as in the stationary case we need the joint semiconcavity in (t, x) of the function $\hat{v}(\cdot, \cdot, a)$. semiconcavity

Hence we need to introduce an additional assumption on the measure $p(t, \mathrm{d}z)$ in order to guarantee it.

Assumption (3): for every $a \in A - \{0\}$, the map

$$(t, x) \longrightarrow \lambda \int w(x + az)p(t, \mathrm{d}z)$$

is (locally) semiconcave for $(t, x) \in (0, +\infty) \times (l(a), +\infty)$ for all measurable continuous functions w with linear growth.

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We have the following.

Theorem

Let Assumptions (1)-(2)-(3) stand in force + local semiconcavity of $\hat{v}$.

- $\forall a \geq 0$ *we have* $\hat{v}(\cdot,\cdot,a) \in C^1([0,+\infty) \times (l(a),+\infty))$ *and*

$$\frac{\partial \hat{v}}{\partial x}(t, l(a)^+, a) = +\infty \text{ for every } t \geq 0.$$

- *We have* $v \in C^1(0,+\infty)$ *and* $v'(0^+) = +\infty$.

4. Existence and characterization of optimal strategies

4.a Feedback representation form of the optimal strategies

Let Assumptions (1)-(2)-(3) stand in force from now on.

Since the viscosity solution to the coupled IPDE is *regular*, we have the following result which provides the optimal control in feedback form.

Verification Theorem

Let $(v, \hat{v})$ be the regular value functions solution to:

$$(\rho + \lambda)\hat{v} - \frac{\partial \hat{v}}{\partial t} - \tilde{U}\left(\frac{\partial \hat{v}}{\partial x}\right) - \lambda \int v(x + az)p(t, \mathrm{d}z) = 0, \quad (t, x, a) \in \mathcal{D}$$

$$v(x) = \mathcal{H}\hat{v}(x) = \sup_{a \in \left[-\frac{x}{\underline{z}}, \frac{x}{\underline{z}}\right]} \hat{v}(0, x, a), \qquad x \geq 0,$$

together with the boundary conditions (G1)-(B1).

Then there exists an optimal control policy (α^*, c^*) given by:

- trading portfolio from the scalar maximum problem

$$\alpha^*_{k+1} = \arg \max_{-\frac{X_k^x}{\bar{z}} \le a \le \frac{X_k^x}{\underline{z}}} \hat{v}(0, X_k^x, a), \quad k \ge 0.$$

- consumption from the deterministic control problem

$$c^*_t = \hat{c}\left(t - \tau_k, Y_t^{(k)}, \alpha^*_{k+1}\right), \quad \tau_k < t \le \tau_{k+1},$$

with

$$\hat{c}(t, x, a) = \arg \max_{c \ge 0} \left[U(c) - c\frac{\partial \hat{v}(t, x, a)}{\partial x}\right] = I\left(\frac{\partial \hat{v}(t, x, a)}{\partial x}\right),$$

where X_k^x is the wealth investor at time τ_k, $Y_t^{(k)} = X_k^x - \int_{\tau_k}^t c^*_s \mathrm{d}s$ and $I = (U')^{-1}$.

4.b Representation of the optimal strategies as the solution of the Euler-Lagrange ODE

From the regularity results, we can deduce more properties of the optimal consumption policy.

- **Stationary case**
  We get an autonomous equation for the optimal consumption policy between two trading dates.

Proposition

Suppose that $U \in C^2((0,+\infty))$ *with* $U''(x) < 0$*, for all* x*. Then the wealth process* Y *between two trading dates is twice differentiable and satisfies the second-order ODE:*

$$\frac{\mathrm{d}^2 Y_t}{\mathrm{d}t^2} = \frac{g'(Y_t, a) - (\rho + \lambda)U'(c_t)}{U''(c_t)}, \quad c_t = -\frac{\mathrm{d}Y_t}{\mathrm{d}t}, \tag{5}$$

where

$$g : \mathcal{X} \longrightarrow \mathbb{R}_+,\ g(x,a) = \lambda \int v(x + az)p(\mathrm{d}z).$$

Case of power utility

In this case, equation (5) takes the form

$$\frac{\mathrm{d}^2 Y_t}{\mathrm{d}t^2} = \frac{\rho+\lambda}{1-\gamma} c_t - \frac{1}{K_1 \gamma (1-\gamma)} c_t^{2-\gamma} g'(Y_t, a), \ Y_0 = x, \ Y_\infty = l(a). \tag{6}$$

Then we can deduce a simple *exponential lower bound* on the integrated consumption, corresponding to the solution of (6) in the case $g \equiv 0$:

$$Y_t \geq Y_t^0, \quad t \geq 0,$$

with

$$Y_t^0 = x - (x - l(a)) \left(1 - e^{-\frac{(\rho+\lambda)t}{1-\gamma}}\right). \tag{7}$$

- **Nonstationary case**
  The regularity results for the optimal strategies are weaker and more difficult to prove.
  As in the S.C., we can deduce an autonomous equation for the optimal wealth process between two trading dates. However, the proof is different and makes use of the Maximum Principle.

Proposition

Let $U \in C^2((0,+\infty))$ *with* $U''(x) < 0$*, for all* x*. Then the optimal wealth process* Y *between two trading dates is twice differentiable and satisfies:*

$$\frac{\mathrm{d}^2 Y_s}{\mathrm{d}s^2} = \frac{\frac{\partial g(s,Y_s,a)}{\partial x} - (\rho+\lambda)U'(c_s)}{U''(c_s)}, \quad c_s = -\frac{\mathrm{d}Y_s}{\mathrm{d}s}, \quad Y_t = x. \tag{8}$$

and $\lim_{t\to+\infty} Y_t = I(a)$.

Case of power utility

Equation (8) becomes:

$$\frac{\mathrm{d}^2 Y_t}{\mathrm{d}t^2} = \frac{\rho+\lambda}{1-\gamma}c_t - \frac{\lambda\theta_1 c_t^{2-\gamma}}{K_1(1-\gamma)}\int (Y_t + az)^{\gamma-1} p(t,\mathrm{d}z), \; Y_0 = x, \; Y_\infty = I(a).$$

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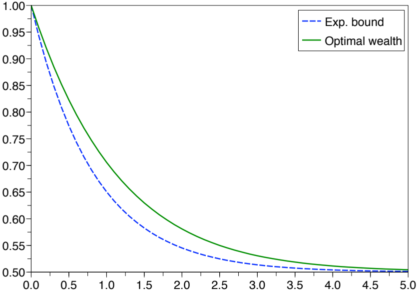
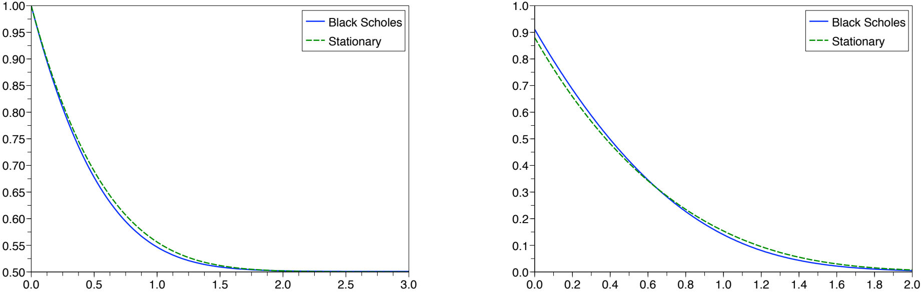
5.a Numerical illustrations: stationary case



Left: typical profile of the optimal wealth process Y_t and the exponential lower bound (7). *Right*: the corresponding consumption strategies. In the presence of investment opportunities, the agent first consumes slowly but if the investment opportunity does not appear, the agent eventually "gets disappointed" and starts to consume fast.

5.b Numerical illustrations: nonstationary case



Optimal wealth (*left*) and consumption policy (*right*) for the probability distribution extracted from the Black-Scholes model (solid line) and from the stationary model having the same distribution as the Black-Scholes model in 3 years' time (dashed line).
Same parameter values as in [PT 07(b)]: drift $b = 0.4$, volatility $\sigma = 1$, discount factor $\rho = 0.2$, intensity $\lambda = 2$ and risk aversion coefficient $\gamma = 0.5$. At least qualitatively, the consumption profile is similar to the one observed in the stationary model, with exponential decay.

Viscosity solutions

In the paper [PT 07(a)] the viscosity solution approach is developed giving existence and uniqueness. Viscosity solutions for the coupled IPDE :

$$
\begin{aligned}
(\rho+\lambda)\hat{v} - \frac{\partial \hat{v}}{\partial t} - \tilde{U}\left(\frac{\partial \hat{v}}{\partial x}\right) - \lambda \int v(x+az)p(t,\mathrm{d}z) = 0, \quad t \geq 0, x > l(a), \\
v(x) \ = \ \mathcal{H}\hat{v}(x) \ = \ \sup_{a \in \left[-\frac{x}{\underline{z}}, \frac{x}{\overline{z}}\right]} \hat{v}(0,x,a), \quad x \in \mathbb{R}_+,
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Definition

Viscosity solutions

In the paper [PT 07(a)] the viscosity solution approach is developed giving existence and uniqueness. Viscosity solutions for the coupled IPDE :

$$(\rho+\lambda)\hat{v} - \frac{\partial \hat{v}}{\partial t} - \tilde{U}\left(\frac{\partial \hat{v}}{\partial x}\right) - \lambda \int v(x+az)p(t,\mathrm{d}z) = 0, \quad t \geq 0, x > l(a),$$

$$v(x) = \mathcal{H}\hat{v}(x) = \sup_{a \in \left[-\frac{x}{z}, \frac{x}{z}\right]} \hat{v}(0,x,a), \quad x \in \mathbb{R}_+,$$

Definition

The pair of functions $(v,\hat{v}) \in C_+(\mathbb{R}_+) \times C_+(\mathcal{D})$ *is a viscosity supersolution to the above IPDE if :*

(i) $v \geq \mathcal{H}\hat{v}$;

(ii) for all $a \in A$, $(\bar{t},\bar{x}) \in \mathbb{R}_+ \times (l(a),\infty)$,

$$(\rho+\lambda)\hat{v}(\bar{t},\bar{x},a) - \frac{\partial \varphi}{\partial t}(\bar{t},\bar{x}) - \tilde{U}\left(\frac{\partial \varphi}{\partial x}(\bar{t},\mathrm{d}\bar{x})\right) - \lambda \int v(\bar{x}+az)p(\bar{t},\mathrm{d}z) \geq 0,$$

for any test function $\varphi \in C^1(\mathbb{R}_+ \times (l(a),+\infty))$, *which is a local minimum of* $(\hat{v}(.,.,a)-\varphi)$.

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Theorem

Under Assumptions (1)-(2), the pair of value functions $(v, \hat{v})$ is the **unique viscosity solution** *to the IPDE :*

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satisfying the boundary conditions (G1)-(B1). BC

Remark

In particular for every $a \in A$, the value function $\hat{v}(\cdot, \cdot, a)$ is the unique viscosity solution of the first HJ equation with the BC (G1)-(B1).

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▶ Idea of the proof

First we prove that $\hat{v}(\cdot, a) \in C^1(I(a), +\infty)$. The other regularity properties follow by a kind of bootstrap argument based on the coupling.

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- Assume by contradiction that $D_x^+\hat{v}(x,a) = [p_1, p_2]$, $p_1 \neq p_2$. Then there exist sequences $x_n,\ y_m \in \mathbb{R}_+$ where $\hat{v}$ is differentiable and s.t.

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Semidifferentials

Definition

Let u be a continuous function on an open set $D \subset \Omega$. For any $y \in D$, the sets

$$D^- u(y) = \left\{ p \in \Omega : \liminf_{z \in D, z \to y} \frac{u(z) - u(y) - \langle p, z - y \rangle}{|z - y|} \geq 0 \right\},$$
$$D^+ u(y) = \left\{ p \in \Omega : \limsup_{z \in D, z \to y} \frac{u(z) - u(y) - \langle p, z - y \rangle}{|z - y|} \leq 0 \right\}$$

are called respectively, the (Fréchet) *subdifferential* and *superdifferential* of u at y.

◂ Return

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Semiconcavity

Definition

We say that a function $u : S \to \mathbb{R}$ is *semiconcave* if there exists a nondecreasing upper semicontinuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{\rho \to 0^+} \omega(\rho) = 0$ and

$$\eta u(x_1) + (1-\eta)u(x_2) - u(\eta x_1 + (1-\eta)x_2) \leq \eta(1-\eta)|x_1 - x_2|\omega(|x_1 - x_2|),$$

for any pair x_1, x_2 such that the segment $[x_1, x_2]$ is contained in S and for $\eta \in [0,1]$. In particular we call *locally semiconcave* a function which is semiconcave on every compact subset of its domain of definition.

Clearly, a concave function is also semiconcave. An important example of semiconcave functions is given by the smooth ones.

Proposition

Let $u \in C^1(A)$, with A open. Then both u and $-u$ are locally semiconcave in A with modulus equal to the modulus of continuity of Du.

Return

3.b Nonstationary case

To use the same argument as in the stationary case we need the joint semiconcavity in (t, x) of the function $\hat{v}(\cdot, \cdot, a)$. semiconcavity

Hence we need to introduce an additional assumption on the measure $p(t, \mathrm{d}z)$ in order to guarantee it.

Assumption (3): for every $a \in A - \{0\}$, the map

$$(t, x) \longrightarrow \lambda \int w(x + az) p(t, \mathrm{d}z)$$

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