The Evaluation of American Compound Option Prices under Stochastic Volatility

Carl Chiarella and Boda Kang

School of Finance and Economics
University of Technology, Sydney

CNR-IMATI Finance Day
Wednesday, 8th April, 2009
Plan of Talk

- Compound options
- A two pass PDE free BVP
- Sparse grid implementation
- Monte Carlo simulation implementation
- Numerical examples
- Conclusions
**Compound Option**

- Compound option is an option on an option. Allow for large leverage. Often used in currency and fixed-income markets.

- **Problem**: the tender for a contract that needs 4 years of financing. If company wins tender, could be exposed to an interest rate rise by the time the contract starts (say 3 months).

- **Solution**: purchase a 3-month call option on a 4 year cap.

- **Properties**: easily leverage a position; with very little upfront premium but possible to take a substantial position.
Figure 1: The components of a compound option
Literature on the Evaluation of Compound Options

• Geske (1979)-the first closed-form solution for the price of a vanilla European call on a European call.

• Han in his thesis (2003) and Fouque and Han (2005):
  – derive a fast, efficient and robust “approximation” to compute the prices of compound options within the context of multiscale stochastic volatility models;
  – they only consider the case of European option on European option; their method relies on certain expansions so its range of validity is not entirely clear.

Compound Options - Evaluation under Stochastic Volatility

- We follow Heston assuming the dynamics for $S$ under RN measure governed by

\[
dS = (r - q)Sdt + \sqrt{v}SdZ_1, \quad (1)
\]

\[
dv = (\kappa_v \theta_v - (\kappa_v + \lambda)v)dt + \sigma \sqrt{v}dZ_2. \quad (2)
\]

- Here $S$ and $v$ are correlated with $E(dZ_1 dZ_2) = \rho dt$.

- Assumes market price of vol. risk $= \lambda \sqrt{v}$. 
The price of an American compound option under SV can be formulated as the solution to a **two-pass Free-BVP** with the following Kolmogorov operator:

\[
\mathcal{K} = \frac{vS^2}{2} \frac{\partial^2}{\partial S^2} + \rho\sigma vS \frac{\partial^2}{\partial S \partial v} + \frac{\sigma^2 v}{2} \frac{\partial^2}{\partial v^2}
\]

\[
+ (r - q) S \frac{\partial}{\partial S} + (\kappa_v (\theta_v - v) - \lambda v) \frac{\partial}{\partial v}.
\]  

(3)
• First the **PDE for the value of the daughter option** \(D(S, v, t)\) with the Kolmogorov operator \(\mathcal{K}\):

\[
\mathcal{K}D - rD + \frac{\partial D}{\partial t} = 0.
\]  

(4)

• Solve on \(0 \leq t \leq T_D\) s.t. \(D(S, v, T_D) = (S - K_D)^+\), and free (early exercise) boundary and smooth pasting conditions:

\[
D(d(v, t), v, t) = d(v, t) - K_D;
\]  

(5)

\[
\lim_{S \to d(v, t)} \frac{\partial D}{\partial S} = 1, \quad \lim_{S \to d(v, t)} \frac{\partial D}{\partial v} = 0.
\]  

(6)

• Here \(S = d(v, t)\) is the early exercise boundary for the daughter option at time \(t\) and variance \(v\).
• Next, the **PDE for the mother option** $M(S, v, t)$:

$$\mathcal{KM} - rM + \frac{\partial M}{\partial t} = 0.$$  \hspace{1cm} (7)

• Solve on $0 \leq t \leq T_M$ with terminal condition

$$M(S, v, T_M) = (D(S, v, T_M) - K_M)^+,$$ \hspace{1cm} (8)

and free (early exercise) boundary and smooth pasting conditions:

$$M(m(v, t), v, t) = D(m(v, t), v, t) - K_M,$$ \hspace{1cm} (9)

$$\lim_{S \to m(v, t)} \frac{\partial M}{\partial S} = \frac{\partial D}{\partial S}, \quad \lim_{S \to m(v, t)} \frac{\partial M}{\partial v} = \frac{\partial D}{\partial v}. \hspace{1cm} (10)$$

• Here $S = m(v, t)$ is the early exercise boundary for the mother option at time $t$ and variance $v$. 
Sparse Grid Implementation

- It is computationally demanding to solve the two nested PDEs (4) and (7). Hence we apply the sparse grid approach.
- We implement the **sparse grid combination technique** of Reisinger and Wittum (2007) to solve these PDEs.
- The technique relies on a combination technique requiring solution of the original equation only on several specific grids and a subsequent extrapolation step.
- In Figure 3, **those grids are dense in one direction but sparse in the other direction**. Solve the two PDEs on each of the grids in parallel and combine the results from different grids.
Figure 2: A typical $S$ (vertical) — $v$ (horizontal) axes. A sparse grid with a level 6 with respect to the combinations (from left to right), (6,0), (5,1), (4,2), (3,3), (2,4), (1,5), (0,6). The (6, 0) grid has $2^6$ subintervals in the $S$— direction and $2^0$ subintervals in the $v$— direction, and so forth.
Figure 3: A typical $S$ (vertical) — $v$ (horizontal) axes. A sparse grid with a level 5 with respect to the combinations (from left to right), (5,0), (4,1), (3,2), (2,3), (1,4), (0,5). The (3, 2) grid has $2^3$ subintervals in the $S$ — direction and $2^2$ subintervals in the $v$ — direction, and so forth.
Following the combination technique, the solution $c_l$ ($l$ sparse grid level) of the PDE is

\[
c_l = \sum_{n=0}^{l} C(l - n, n) - \sum_{n=0}^{l-1} C(l - 1 - n, n). \quad (11)
\]

The combination gives a more accurate solution of the PDE.

There are $(2l + 1)$ PDE solvers running in parallel at the same time on each of the sparse grids for level $l$ and level $(l - 1)$ respectively.

Because of different scale characteristics in $S$ and $v$ direction and also some bad behavior along the boundary we need to use a modified sparse grid. (see Figure 4)
Figure 4: A **modified sparse grid** with a initial level 2 and total level 6 with respect to each combination. From left to right we see the combinations (2,4), (3,3), (4,2).
Monte Carlo simulation implementation

- We need an alternative method to check the solution.
- Use the Method of Lines (MOL) to solve the PDE for the Daughter option and obtain the option prices with a range of maturities, and store the results.
- Implement Monte Carlo Simulation scheme of Ibanez & Zapatero (2004) to find the price of the American mother option with suitable terminal condition (8) and free boundary condition (9).
- The data of the underlying daughter option are available from the previous results from MOL.
Figure 5: Solving for the free boundary point of the Daughter option along a \((v_m, \tau_n)\) line using MOL.
Figure 6: Illustrating the MOL-MC scheme along one \((S, v)\) line
Monte Carlo Simulation for the Mother option.

- A finite number of exercise opportunities $0 = t_0 < t_1 \cdots < t_N = T_M$ are considered.
- The optimal exercise strategy at every point at time $t_n$ is characterized by a region in a two dimensional-space $(v, t_n)$.
- Going backward in time, we solve the following equation for each $n = N - 1, \ldots, 1$ at different variance levels $v_i$:

$$M(S, v_i, t_n) = D(S, v_i, t_n) - K_M;$$

for $S$ using Newton’s method to find the optimal exercise frontier $S^*_{t_n}(v_i)$;
Figure 7: Monte Carlo Simulation for the Mother option for some fixed $\nu$
– Continuing to work backwards, we can find all optimal strategies at times $t_0 < t_1, \cdots, < t_N$ for a certain number of variance levels.

– Finally implementing another MC simulation to generate paths for both the underlying prices and the variance forward in time starting from $t_0$ to find the price of the mother option $M(S, v, t_0)$ based on all known optimal exercise strategies.
### Numerical Examples

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>SV Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.03</td>
<td>$\theta$</td>
<td>0.04</td>
</tr>
<tr>
<td>$q$</td>
<td>0.05</td>
<td>$\kappa_v$</td>
<td>2.00</td>
</tr>
<tr>
<td>$T_D$</td>
<td>1.0</td>
<td>$\sigma$</td>
<td>0.20</td>
</tr>
<tr>
<td>$K_D$</td>
<td>100</td>
<td>$\lambda_v$</td>
<td>0.00</td>
</tr>
<tr>
<td>$T_M$</td>
<td>0.60</td>
<td>$\rho$</td>
<td>±0.50</td>
</tr>
<tr>
<td>$K_M$</td>
<td>7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Parameter values used for the American call daughter option. The stochastic volatility (SV) parameters correspond to the Heston model.
$\rho = -0.50, \nu = 0.04$

<table>
<thead>
<tr>
<th>Method</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>Runtime (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SG (4,6)</td>
<td>0.0769</td>
<td>0.6922</td>
<td>2.9632</td>
<td>7.7054</td>
<td>14.7299</td>
<td>1510</td>
</tr>
<tr>
<td>MOL + MC (500,000)</td>
<td>0.0758</td>
<td>0.6898</td>
<td>2.9567</td>
<td>7.6920</td>
<td>14.7089</td>
<td>2997</td>
</tr>
<tr>
<td>std err</td>
<td>0.0006</td>
<td>0.0019</td>
<td>0.0041</td>
<td>0.0065</td>
<td>0.0078</td>
<td></td>
</tr>
<tr>
<td>Lower Bound</td>
<td>0.0747</td>
<td>0.6860</td>
<td>2.9486</td>
<td>7.6793</td>
<td>14.6937</td>
<td></td>
</tr>
<tr>
<td>Upper Bound</td>
<td>0.0770</td>
<td>0.6935</td>
<td>2.9649</td>
<td>7.7046</td>
<td>14.7241</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Compound prices (American call on American call) computed using sparse grid (SG), Monte Carlo simulation (MC) together with method of lines (MOL). Parameter values are given in Table 1, with $\rho = -0.50$ and $\nu = 0.04$. 
$\rho = 0.50, v = 0.04$

<table>
<thead>
<tr>
<th>Method</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>Runtime (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SG (4,6)</td>
<td>0.3124</td>
<td>1.1945</td>
<td>3.4102</td>
<td>7.6791</td>
<td>14.2885</td>
<td>1495</td>
</tr>
<tr>
<td>MOL + MC (500,000)</td>
<td>0.3128</td>
<td>1.1940</td>
<td>3.4061</td>
<td>7.6646</td>
<td>14.2716</td>
<td>2950</td>
</tr>
<tr>
<td>std err</td>
<td>0.0015</td>
<td>0.0030</td>
<td>0.0050</td>
<td>0.0070</td>
<td>0.0080</td>
<td></td>
</tr>
<tr>
<td>Lower Bound</td>
<td>0.3099</td>
<td>1.1882</td>
<td>3.3964</td>
<td>7.6509</td>
<td>14.2558</td>
<td></td>
</tr>
<tr>
<td>Upper Bound</td>
<td>0.3157</td>
<td>1.1998</td>
<td>3.4159</td>
<td>7.6783</td>
<td>14.2873</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Compound prices (American call on American call) computed using sparse grid (SG), Monte Carlo simulation (MC) together with method of lines (MOL). Parameter values are given in Table 1, with $\rho = 0.50$ and $v = 0.04$. 
Figure 8: Free surfaces of both daughter option and mother option, with the parameters in Table 1 and $\rho = -0.5$. 
Figure 9: Free surfaces of both daughter option and mother option, with the parameters in Table 1 and $\rho = 0.5$. 
Figure 10: Free surfaces of daughter option with different $\rho$, the time to maturity $\tau$ and parameters in Table 1.
Figure 11: Free surfaces of mother option with different $\rho$, the time to maturity $\tau$ and parameters in Table 1.
Figure 12: Free surfaces of daughter option with different $\rho$, the variance $\nu_0$ and parameters in Table 1.
Figure 13: Free surfaces of mother option with different $\rho$, the variance $\nu_0$ and parameters in Table 1.
Figure 14: Percentage price differences between constant and stochastic volatility $\rho < 0$. 
Figure 15: Percentage price differences between constant and stochastic volatility $\rho > 0$. 
Conclusions

- Compound Options under Stochastic Volatility
- Allow for early exercise feature
- Set up as two pass PDE FBVP
- Solve using the sparse grid technique
- Benchmark against MOL/MC
- The sparse grid approach can be speeded up by using better PDE solvers, e.g. MOL, operator splitting
- Future work; apply to specific examples e.g. real options applications such as multi-stage investment projects