Stability, permanence and optimal control of a Holling-II type trophic system with two sources and pulses

Luca Galbusera and Sara Pasquali *

Abstract

In this report, we study a multi-trophic prey-predator system composed by top- and intermediate-level consumer and two food sources. Biomass fluxes across trophic levels are modeled by means of Holling-II type functional responses and one of the sources is subject to periodic impulsive events consisting in biomass injections. Referring to this system, we analyze the stability properties of periodic eradicated solutions in terms of relations between the model parameters and the intensity of the impulses, which affect the permanence of the species. Secondly, we formulate an optimal control problem for this system as well as an Iterative Dynamic Programming (IDP) scheme for its solution. By means of the proposed algorithm we specify the intensity of the impulsive controls to be applied periodically on the system in order to regulate the population levels along time.

Keywords: trophic system; stability; optimal control; impulse.

1 Introduction

In the recent literature, significant effort has been devoted to the study of optimal natural resource exploitation strategies for human welfare and economic growth [1], [2], especially in contexts where the role of human presence is explicitly taken into account [3], [4], [5], [6]. Control theory, in particular, has been employed in contexts such as harvesting [7], [8], fishery management [9], [10], [11], pest control [12], [13], [14], [15]. Various classes of trophic chains have been considered, see for instance: [16] for optimal harvesting in single growing species; [17] for stabilization and synchronization problems in Lotka-Volterra models; [18] for optimal foraging in adaptive predator-prey systems; [19] and [20], applying optimal control to a Rosenzweig-MacArthur tritrophic system in order to achieve sustainable management strategies with either top-down and bottom-up control setups; [21], accounting for various types of functional responses used to describe predator-prey interactions.

On the other side, in recent studies on trophic systems impulses are often used to represent fast biomass injections at some levels of a given trophic chain or sudden events such as the birth of new individuals within a population. Some representative works include: [22], dealing with a tritrophic Lotka-Volterra chain with pulses on the intermediate-level predator; [23], based on a tritrophic system with Holling-IV type functional responses and pulses acting on the top predator biomass; [24], studying a system with pulses on both prey and predator populations.

---

*CNR, Istituto di Matematica Applicata e Teconologie Informatiche “Enrico Magenes”, Via Bassini 15, 20133 Milano, Italy. Email addresses: galbusera@mi.imati.cnr.it and sara.pasquali@mi.imati.cnr.it.
local and global stability of periodic solutions and permanence conditions for the species; [25],
dealing with a Lotka-Volterra system with competing species. See also references [26], [27], [28]
and [29] for recent developments.

The main focus of this report is on the analysis and optimal impulsive control of a trophic
system subject to periodic impulsive events and endowed with a special structure, which differs
from those most frequently considered in the previous literature. It comprises two food sources,
intermediate- and top-level consumers, whose interaction is described by Holling-II type models.
The first food source is accessed by the intermediate consumers, while the second one is exploited
exclusively by the top-level consumers. A real-world example of such kind of system is represented
by a trophic system with two plant sources representing the pasture and the crop periodically
stocked in warehouses, respectively. In this example, the growth of the pasture will be represented
as a continuous process along time. On the other side, crop stock undergoes a periodic filling-
depletion cycle, determined by the agricultural cycle making its products available according to
a specified seasonality.

Considering this system, we will first discuss its structural properties, including the bounded-
ness of its trajectories, and stability properties of specific periodic solutions of the system which
influence the permanence of all species. Finally, we will formulate an optimal control problem
for this system, wherein the control action consists in the choice of the intensity of the periodic
impulsive biomass injections. We will show how the presence of the crop stock, subject to pulses,
can be exploited in order to regulate the population levels along time. We will also discuss an
algorithm for the solution of such a control problem, based on the concept of Iterative Dynamic
Programming [30].

The report is organized as follows: in Section 2 we introduce the trophic model under anal-
alysis; Section 3 summarizes some of its basic structural properties and introduces the periodic
eradicated solutions of the system; their local stability properties and their relationship with
permanence are discussed in Section 4. Finally, in Section 5 we formulate an associated opti-
mal control scheme and propose a numerical algorithm for its solution, applied to a numerical
example.

**Notation:** \(\mathbb{N}\) is the set of naturals and \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{R}_+ = [0, \infty)\) and \(\mathbb{R}^+_0 = (0, \infty), \mathbb{R}^+_1 =
\{x \in \mathbb{R}_+^4 : x \geq 0\}\) and \(\mathbb{R}^+_2 = \{x \in \mathbb{R}_+^4 : x > 0\}\), \(PC(\mathbb{R}_+, \mathbb{R}) (PC^1(\mathbb{R}_+, \mathbb{R}))\) the class of real piecewise
continuous (continuously differentiable) functions defined on \(\mathbb{R}_+\). Symbol \(\left(\cdot\right)^T\) denotes transpose, \(\dot{x}(t)\) the
time derivative of \(x\) and \(mod(\cdot)\) the modulo function.

## 2 Model formulation

Define a countable set \(T\) composed of fixed time instants \(t_k \in \mathbb{R}_+, k \in \mathbb{N}_0\) such that \(t_0 = 0, \ t_{k+1} > t_k, \ \forall k\) and \(\lim_{k \to \infty} t_k = \infty\). Furthermore, introduce the following trophic chain model:

\[
\begin{align*}
\dot{x}_1(t) &= r_1 x_1(t) g\left(\frac{x_1(t)}{K_1}\right) - D_{13} f_{31}(x_1(t)) x_3(t) \\
\dot{x}_2(t) &= -q_2 x_2(t) - D_{24} f_{42}(x_2(t)) x_4(t), \ \forall t \in \mathbb{R}_+ \setminus T \\
x_2(t_k^+) &= x_2(t_k) + h(t_k), \ \forall k \in \mathbb{N}_0 \\
\dot{x}_3(t) &= \left[D_{31} f_{31}(x_1(t)) - q_3\right] x_3 - D_{34} f_{43}(x_3(t)) x_4(t) \\
\dot{x}_4(t) &= \left[D_{42} f_{42}(x_2(t)) + D_{43} f_{43}(x_3(t)) - q_4\right] x_4(t)
\end{align*}
\]

with initial condition \(x(0) = x^0\), where \(x = [x_1, x_2, x_3, x_4]^T\) and \(x^0 = [x_1^0, x_2^0, x_3^0, x_4^0]^T\) is non-
negative. State variables represent biomasses, with \(x_1\) and \(x_2\) referring to the two food sources,
$x_3$ to the intermediate-level consumer and $x_4$ to the top-level consumer. Parameter $r_1 > 0$ is the specific growth rate of the first source, $K_1 > 0$ its carrying capacity and

$$g \left( \frac{x_1}{K_1} \right) = 1 - \frac{x_1}{K_1} \quad (2)$$

For all $(i, j) \in \{(1, 3), (2, 4), (3, 4)\}$, trophic interactions are modeled by means of a Holling-II type functional response, i.e.

$$f_{ji}(x_i) = \frac{x_i}{a_{ji} + x_i} \quad (3)$$

where $a_{ji} > 0$. The biomass flow across levels also depends on the positive constants $D_{ij}$, representing the food demands of consumers per time instant. Relationship $D_{ji} = \theta_{ji} D_{ij}$ holds, where $\theta_{ji} \in (0, 1]$ represents the efficiency of the biomass conversion associated to each trophic process. Quantities $q_2, q_3, q_4 > 0$ are specific loss rates. In view of (3), we assume $D_{31} - q_3 > 0$ and $D_{42} + D_{43} - q_4 > 0$, which are necessary conditions to avoid trivial dynamics in the ODEs associated to $x_3$ and $x_4$, respectively.

The second food source is accessed exclusively by the top-level consumer and is subject to externally-forced pulses $h(t_k) \geq 0$, representing the biomass injected into the second trophic level at time $t_k \in \mathcal{T}$. In this sense, $q_2$ represents the deterioration rate of the stocked biomass. From now on in this report we assume

$$t_k - t_{k-1} = P, \ k \in \mathbb{N} \quad (4)$$

where $P > 0$ is a constant. Periodic impulsive events are referred to as they can can represent significant classes of real-world situations, as the one in which crop becomes periodically available due to the natural cycle and is then stocked. Furthermore, for analysis purposes, in most of Sections 3-4 we will also assume

$$h(t_k) = \bar{h}, \ k \in \mathbb{N} \quad (5)$$

where $\bar{h} > 0$ is a constant. Instead, in the last part of this report we will refer to time-varying impulse intensities, for control purposes.

### 3 Invariant sets, solution boundedness and periodic eradicated solutions system

Thanks to Lemma 1 in [29], we can prove the next two statements about system (1):

**Lemma 1** $\mathbb{R}^{4+}$ is invariant for system (1).

**Proof:** Consider a solution of system (1) with initial condition $x^0 \in \mathbb{R}^{4+}$. In view of (2), assuming in particular that $x_1^0 \leq K_1$, it results $\dot{x}_1(t) \geq p_1(t)x_1(t)$, with $p_1(t) = -D_{13}a_{31}^{-1}x_3(t)$. Furthermore, we have $\dot{x}_2(t) \geq p_2(t)x_2(t), \forall t \in \mathbb{R}^+ \setminus \mathcal{T}$ with $p_2(t) = -q_2 - D_{24}a_{42}^{-1}x_4(t)$, $\dot{x}_3(t) \geq p_3(t)x_3(t)$ with $p_3(t) = D_{31}f_{31}(x_1(t)) - q_3 - D_{43}a_{43}^{-1}x_4(t)$ and $\dot{x}_4(t) = p_4(t)x_4(t)$ with $p_4(t) = D_{42}f_4(x_2(t)) + D_{43}f_{43}(x_3(t)) - q_4$. In all these cases, since $x_i(t) \geq x_i^0 e^{\int_0^t p_i(\theta)d\theta}, \forall t \in \mathbb{R}_+ \setminus \mathcal{T}$, and $h(t_k) \geq 0, \forall k \in \mathbb{N}_0$, the proof is complete.

**Lemma 2** All solutions $x(t)$ of (1) subject to (5) and with $x^0 \in \mathbb{R}^{4+}$ are bounded.
Proof: Consider a solution $x(t)$ of system (1) with $x^0 \in \mathbb{R}^4_+$ and define

$$V(x) = \theta_{43} \theta_{31} x_1 + \theta_{42} x_2 + \theta_{43} x_3 + x_4$$

(6)

For $t \in \mathbb{R}_+ \setminus \mathcal{T}$ it results

$$\dot{V}(x(t)) = \theta_{43} \theta_{31} r_1 x_1(t) g \left( \frac{x_1(t)}{K_1} \right) - \theta_{42} q_2 x_2(t) - \theta_{43} g_3 x_3(t) - q_4 x_4(t)$$

Now define $q_m = \min_{i=2,3,4} q_i$. It is easy to verify that, $\forall t \in \mathbb{R}_+ \setminus \mathcal{T}$,

$$\dot{V}(x(t)) + q_m V(x(t)) \leq \theta_{43} \theta_{31} \left[ r_1 g \left( \frac{x_1(t)}{K_1} \right) + q_m \right] x_1(t)$$

and consequently, in view of (2),

$$\dot{V}(x(t)) + q_m V(x(t)) \leq C$$

(7)

with $C = \theta_{43} \theta_{31} \frac{(r_1 + q_m)^2 K_1}{4 r_1}$. On the other side, $\forall k \in \mathbb{N}_0$,

$$V(x(t_k^+)) = V(x(t_k)) + \theta_{42} \bar{h}$$

(8)

Finally, formulas (7) and (8) imply

$$V(x(t)) \leq V(x(0^+)) e^{-q_m t} + C \frac{1 - e^{-q_m t}}{q_m} + \sum_{0 < t_k < t} e^{-q_m (t-t_k)} \bar{h}$$

(9)

In view of the latter inequality, it results that $V$ is bounded and $x$ is also bounded as a consequence.

The next corollary about the ultimate boundness of the solutions of (1) follows immediately from (6) and (9):

**Corollary 1** The solutions of (1) subject to (4) and (5) fulfill

$$\limsup_{t \to \infty} x_1(t) \leq (\theta_{43} \theta_{31})^{-1} G,$$

$$\limsup_{t \to \infty} x_2(t) \leq (\theta_{42})^{-1} G,$$

$$\limsup_{t \to \infty} x_3(t) \leq (\theta_{43})^{-1} G$$

and

$$\limsup_{t \to \infty} x_4(t) \leq G,$$

where $G = \frac{C}{q_m} + \frac{\bar{h}}{1 - e^{-q_m \bar{r}}}$. 

To complete this section, we deal with the existence of periodic solutions for the eradicated forms of system (1) subject to (4) and (5), here defined as the ones in which at least one species among 1, 3 and 4 is extinct.

**Theorem 1** System (1) subject to (4) and (5) admits the following periodic eradicated solutions:

$$\mathcal{S}^0: x^0(t) = (0, \bar{x}_2(t), 0, 0)$$

$$\mathcal{S}^1: x^1(t) = (K_1, \bar{x}_2(t), 0, 0)$$

and, assuming $D_{31} f_{31}(K_1) - q_3 > 0$ (equivalently $1 - \frac{q_3 q_{31}}{(D_{31} - q_3) K_1} > 0$),

$$\mathcal{S}^{13}: x^{13}(t) = (\bar{x}_1^{13}, \bar{x}_2(t), \bar{x}_3^{13}, 0)$$

where

$$\bar{x}_2(t) = \bar{x}_2^0 e^{-q_2 t} \mod \bar{r}, \quad \bar{x}_2^0 = \frac{\bar{h}}{1 - e^{-q_2 \bar{r}}}$$

(10)

and $\bar{x}_1^{13} = \frac{q_3 q_{31}}{D_{31} - q_3}, \bar{x}_3^{13} = \frac{q_3 q_{31}}{(D_{31} - q_3) K_1} \left( 1 - \frac{q_3 q_{31}}{(D_{31} - q_3) K_1} \right)$. Furthermore, if $x_2^0 > 0$, then $\lim_{t \to \infty} |x_2(t) - \bar{x}_2(t)| = 0$. Finally, $x_2(t) \geq \bar{x}_2(t)$ if $x_2^0 \geq \bar{x}_2^0$ and $x_2(t) < \bar{x}_2(t)$ if $x_2^0 < \bar{x}_2^0$. 


Proof: Referring to either $S^0$, $S^1$ or $S^{13}$, the time evolution of the state variable $x_2$ in (1) is governed by equations

$$\begin{align*}
\dot{x}_2 &= -q_2 x_2, \quad \forall t \in \mathbb{R}_+ \setminus T \\
x_2(t^+_k) &= x_2(t_k) + \bar{h}, \quad \forall k \in \mathbb{N}_0
\end{align*}$$

(11)

Thus, $\forall t, t_0 \in (t_k, t_{k+1}]$ and $\forall k \in \mathbb{N}_0$ it results $x_2(t) = e^{-q_2(t-t_0)}x_2(t_0)$ and, by assumption (4), $x_2(t^+_k) = x_2(t_{k+1}) + \bar{h} = e^{-q_2T}x_2(t^+_k) + \bar{h}$. Imposing $x_2(t^+_k) = x_2(t^+_k)$, $\forall k \in \mathbb{N}_0$, we get the periodicity condition $x_2(0^+) = \frac{\bar{h}}{1 - e^{-q_2T}}$, from which (10) follows naturally. As for the second part of the theorem’s statement, observe that the solution of (11) is given by $x_2(t) = (x_2(t^+_k) - \frac{\bar{h}}{1 - e^{-q_2T}}) e^{-q_2(t-t_k)} + \bar{x}_2(t)$ for $t \in (t_k, t_{k+1}]$, $\forall k \in \mathbb{N}_0$. Hence $\lim_{t \to \infty} |x_2(t) - \bar{x}_2(t)| = 0$ and $x_2(t) \geq \bar{x}_2(t)$ if $x_2(0) \geq \bar{x}_2^0$, while $x_2(t) < \bar{x}_2(t)$ if $x_2(0) < \bar{x}_2^0$.

Observe that system (1) with $x_2 = x_4 = 0$ is a Rosenzweig-MacArthur model which can admit limit cycles, see for instance [31]; this topic is beyond the scope of our analysis.

4 Local stability of the periodic eradicated solutions and permanence

We now discuss the local stability properties of the periodic eradicated solutions reported in Theorem 1. To this end, introduce the following variables $\xi_i$, $i = 1, \ldots, 4$, in order to study small-amplitude perturbations on the periodic eradicated solutions:

$$\begin{align*}
\xi_1(t) &= x_1(t) - \bar{x}_1, \quad \xi_2(t) = x_2(t) - \bar{x}_2(t), \quad \xi_3(t) = x_3(t) - \bar{x}_3, \quad \xi_4(t) = x_4(t)
\end{align*}$$

Here the constants $\bar{x}_i$, $i = 1, 3$ and the periodic function $\bar{x}_2(t)$ may represent any of the periodic eradicated solutions discussed above. Substituting the latter relationships in system (1) and linearizing it around $(\bar{x}_1, \bar{x}_2(t), \bar{x}_3, 0)$ leads to the following system:

$$\dot{\xi}(t) = M(t)\xi(t)$$

(12)

where

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}, \quad M(t) = \begin{bmatrix}
m_{11} & 0 & m_{13} & 0 \\ 0 & m_{22} & 0 & m_{24} \\ m_{31} & 0 & m_{33} & m_{34} \\ 0 & m_{42} & m_{43} & m_{44} \end{bmatrix}$$

with $m_{ij} = -D_{ij} \frac{\bar{x}_i}{a_{ji} + \bar{x}_i}$ for $i < j$, $m_{ij} = D_{ij} \frac{a_{ij}}{a_{ji} + \bar{x}_i} \bar{x}_i$ for $i > j$, $m_{11} = r_1 \left(1 - 2 \frac{\bar{x}_1}{K_1}\right) - \theta_1^{-1}m_{31}$, $m_{22} = -q_2 - \theta_2^{-1}m_{42}$, $m_{33} = -\theta_3^{-1}m_{13} - q_3 - \theta_3^{-1}m_{13}$ and $m_{44} = \theta_4^{-1}m_{24} - \theta_4^{-1}m_{34} - \theta_4^{-1}m_{34} - q_4$. Observe that $m_{24}, m_{42}$ and $m_{44}$ vary with time, since they depend on $\bar{x}_2(t)$. Also notice that model (12) is not impulsive, thanks to the properties of the $P$-periodic function $\bar{x}_2(t)$.

Defining $\nu_1 = \frac{q_1}{D_{12}^2}$ and $\nu_1 = \frac{q_4}{D_{42}^2} - \frac{D_{42}^3}{D_{12}^2(a_{43} + \bar{x}_3)}$, we can introduce the following theorem:

Theorem 2 The periodic eradicated solutions $\bar{S}^0$, $\bar{S}^1$ and $\bar{S}^{13}$ of system (1) subject to (4) and (5) fulfill the following local stability properties:

$\bar{S}^0$ is unstable.
\( S^1 \) is LAS (locally asymptotically stable) iff

\[
D_{31} f_{31}(K_1) - q_3 < 0
\]

and

\[
\tilde{h} < a_{42} \left( \frac{e^{\nu_1 q_2 P} - 1}{1 - e^{(\nu_1-1)q_2 P}} \right), \text{ if } \nu_1 < 1
\]

(13)

\( S^{13} \), which exists when \( D_{31} f_{31}(K_1) - q_3 > 0 \), is LAS iff

\[
\begin{cases}
m_{11} + m_{33} < 0 \\
m_{11} m_{33} - m_{13} m_{31} > 0
\end{cases}
\]

and

\[
\tilde{h} < a_{42} \left( \frac{e^{\nu_{13} q_2 P} - 1}{1 - e^{(\nu_{13}-1)q_2 P}} \right), \text{ if } \nu_{13} < 1
\]

(14)

(15)

Proof: see the Appendix.

To conclude this section, consider the following definition of permanence for system (1):

**Definition 1** System (1) is said to be permanent if, for any solution with \( x(0) > 0 \), there exist constants \( \bar{t} > 0 \) and \( x_{\text{min}}, x_{\text{max}} \in \mathbb{R}^4_+ \) such that \( x_{\text{min}} \leq x(t) \leq x_{\text{max}}, \forall t \geq \bar{t} \).

Observe that permanence of system (1) excludes the asymptotic stability of the periodic eradicated solutions reported above.

5 Optimal control

In this section we discuss an optimal impulsive control problem for system (1). In particular, in Subsection 5.1 we formulate the problem with an algorithm for its numerical solution, while in Subsection 5.2 we propose an example.

5.1 Problem formulation and numerical algorithm

Our objective will be to exploit the presence of the second biomass source in order to control the biomasses \( x_1, x_3 \) and \( x_4 \) along time by means of an impulsive control action. In particular, assuming \( h = h(t_k), \forall t_k \in \mathcal{T} \) (with \( h(t_0) = 0 \)), we define the following optimal control problem:

\[
\min_{h(t_k), \forall k \in [1, n-1]} J(x(t_2), \ldots, x(t_n), h(t_1), \ldots, h(t_{n-1}))
\]

s.t. (1) and \( h(t_k) \in [0, h_{\text{max}}], \forall k \in [1, n-1] \)

(17)

with \( J = \sum_{k=1}^{n-1} \left[ \sum_{i=1,3,4} \alpha_i (x_i(t_{k+1}) - x_i^*)^2 + \beta h^2(t_k) \right] \). In this formulation, \( x_i^* \geq 0, i = 1, 3, 4 \) are reference values for the respective state variables, \( \alpha_i \geq 0 \) and \( \beta \geq 0 \) are weights, \( n > 0 \) is the number of periods considered in the optimization problem and \( h_{\text{max}} > 0 \) is a fixed upper bound to the value of the impulses.

In order to solve the optimal control problem (17), we propose an algorithm consisting in dividing the time horizon \([0, nP]\) in stages separated by the impulsive events. The solution is
searched for by means of the algorithm defined by the following steps, which represents an ad-hoc formulation of the Iterative Dynamic Programming (IDP) concept proposed in [30] and based on Bellman's principle of optimality.

1. Fix positive constants $N_1$ and $N_2$ and the feasible set $H_k = [0, h_{\text{max}}]$ for $h(t_k), \forall k \in [1, n]$.

2. Choose $N_1$ values for $h(t_k)$ in $H_k, \forall k \in [1, n-1]$, integrate system (1) stage by stage along the time horizon and store the $N_1$ matrices containing the states $x(t_k^+), k \in [0, n]$ associated to each impulsive input sequence.

3. Referring to stage $n$, integrate system (1) starting from any of the $N_1$ values of $x(t_{n-1}^+)$ found at step 2 and compute the cost-to-go $J_n(j) = \sum_{i=1,3,4} \alpha_i (x_i(t_n) - x_i^*)^2 + \beta h^2(t_{n-1}), \forall j \in [1, N_1]$.

4. Step backward to $n-1$, choose $N_2$ values for $h(t_{n-1})$ in $H_{n-1}$ and integrate (1) starting from any of the $N_1$ values of $x(t_{n-2}^+)$ computed at step 2 until $t_{n-1}^+$ correspondingly. Select $l \in [0, N_1]$ indexing the point $x(t_{n-1}^+)$ resulting from step 2 closest to the state value resulting from integration as an approximate trajectory continuation. Finally, for each $j \in [1, N_1]$ select the impulse value and trajectory continuation minimizing the cumulative cost-to-go $J_n-1(j) = J_n(l) + \sum_{i=1,3,4} \alpha_i (x_i(t_{n-1}) - x_i^*)^2 + \beta h^2(t_{n-2})$.

5. Iterate backwards in time according to the previous step and complete the calculation of the solutions until $t_0 = 0$. Store the values of $h(t_k), \forall k \in [1, n]$ producing the trajectory with the minimum cumulative cost-to-go.

6. $\forall k \in [1, n]$, narrow the impulse intensity range $H_k \subseteq [0, h_{\text{max}}]$ while ensuring that it includes the best value for $h(t_k)$ stored at the previous step.

7. Return to step 2 and iterate until the maximum number of iterations allowed is reached.

5.2 Numerical example

Consider system (1) with the following parameterization: $r_1 = 0.1$, $K_1 = 0.1$, $q_2 = 0.004$, $q_3 = 0.01$, $q_4 = 0.002$, $D_{13} = D_{24} = D_{34} = 0.20$, $\theta_{31} = \theta_{42} = \theta_{43} = 0.20$, $a_{31} = a_{12} = a_{43} = 1$. It is possible to verify that system (1) admits the following coexistence equilibrium for the species $1$, $3$ and $4$ alone: $\tilde{x}_1 = 0.7798$, $\tilde{x}_3 = 0.2501$ and $\tilde{x}_4 = 0.0923$. Fix $x_i(0) = \tilde{x}_i, i = 1, 3, 4$ and $x_2(0) = 0.001$ and consider the optimal control problem (17) with $x_i^* = \tilde{x}_i, i = 1, 3$ and $x_4^* = 0.25$, assuming that our objective is to suitably increment the biomass level of the top-level consumer with a limited displacement of the biomass quantities of levels $1$ and $3$ with respect to the equilibrium values specified above. In (17), we assume $n = 10$ and $h_{\text{max}} = 1$ and choose weights $\alpha_1 = \alpha_3 = 0.01$, $\alpha_4 = 10$ and $\beta = 0.1$, where the comparatively high value of $\alpha_4$ is intended to incentivize the proximity of $x_4$ to $x_4^*$.

By applying the IDP method described above we obtain the results reported in Figure 1. It can be observed that the resulting impulsive strategy enhances the level of biomass $4$, while biomass $3$ is decreases moderately. Also notice that, under the considered parameter setup, the regime values found for $h(t_k)$ by the algorithm exclude the local asymptotic stability of the periodic eradicated solutions of the system, leading to permanence of all the species.
Conclusions

In this report, we studied a multi-trophic system composed of two sources, an intermediate- and an top-level consumer and subject to periodic impulses acting on the second source’s state variable. We analyzed some key features of the proposed system and discussed stability and permanence issues. Furthermore, we proposed an optimal control problem wherein the control variable is represented by the intensity of the impulsive biomass injections taking place periodically. We also discussed a numerical algorithm for its solution, which is based on the concept of Iterative Dynamic Programming.

Appendix

Proof of Theorem 2. In view of the Floquet theorem reported in [32], the local stability properties of the periodic eradicated solutions reported in the theorem statement can be assessed by means of Floquet multipliers, consisting in the eigenvalues of the monodromy matrix $\Phi(P)$ such that $\xi(t + P) = \Phi(P)\xi(t)$ in system (12). We address the different cases separately:

1. $S^0$: $M(t)$ is upper triangular and consequently the corresponding monodromy matrix $\Phi^0(t)$ has the same structure, with $diag(\Phi^0(t)) = (e^{r_1 t}, e^{-q_2 t}, e^{-q_3 t}, e^{\int_0^t D_{42} f(\bar{x}_2(\tau)) - q_4 d\tau})$. Denote by $\lambda_{0,i}$ the eigenvalues of $\Phi^0(P)$, for $i = 1, \ldots, 4$. It is easy to observe that $\lambda_{0,1} > 1$, which implies the instability of the periodic solution of $S^0$.

2. $S^1$: $M(t)$ is upper triangular and the corresponding monodromy matrix $\Phi^1(t)$ has $diag(\Phi^1(t)) = (e^{-r_1 t}, e^{-q_2 t}, e^{(D_{31} f_{31}(K_1) - q_3)t}, e^{\int_0^t D_{42} f(\bar{x}_2(\tau)) - q_4 d\tau})$. We proceed similarly to the previous case, evaluating the associated eigenvalues $\lambda_{1,i}$, $i = 1, \ldots, 4$ of $\Phi^1(P)$. In particular, it results $\lambda_{1,1}, \lambda_{1,2} < 1$. Furthermore, $\lambda_{1,3} < 1$ iff condition (13) holds. As for term $\lambda_{1,4}$, we can proceed

Figure 1: Optimal states and impulsive sequence.
as follows:

\[ \lambda_{1,4} = e^{P} \int_{0}^{P} D_{42} f(\bar{x}_2(\tau)) - q_4 d\tau < 1 \iff \int_{0}^{P} D_{42} f(\bar{x}_2(\tau)) - q_4 d\tau < 0 \]

\[ \iff \bar{x}_2^0 \left( 1 - e^{(\nu_1 - 1)q_2 P} \right) < a_{42} (e^{\nu_1 q_2 P} - 1) \]

from which (14) follows by studying the feasibility of the last inequality.

3. \( \bar{S}^{13} \): in this case, \( M(t) \) has \( m_{22} = -q_2, \ m_{33} = -\theta_{31} m_{13} - q_3 \) and \( m_{42}, \ m_{43} = 0 \). Thus, the characteristic polynomial associated to the monodromy matrix \( \Phi^{13}(P) \) is

\[ \mathcal{P}(\Phi^{13}(P)) = (-\lambda + m_{22}) (-\lambda + m_{44}) (\lambda^2 - (m_{11} + m_{33}) \lambda + (m_{11} m_{33} - m_{13} m_{31})) \]

Observe that \( \lambda_{13,1} = m_{22} = -q_2 < 0 \), while term \( \lambda^2 - (m_{11} + m_{33}) \lambda + (m_{11} m_{33} - m_{13} m_{31}) \) has negative roots iff the Routh-Hurwitz conditions (15) hold. Finally, since term \(-\lambda + m_{44}\) depends on the periodic signal \( \bar{x}_2(t) \), it is necessary to evaluate the corresponding term in the monodromy matrix \( \Phi^{13}(t) \), i.e., \( \phi^{13}_{44} = e^{P} \int_{0}^{P} D_{42} f_{42}(\bar{x}_2(\tau)) + D_{43} f_{43}(\bar{x}_3(\tau)) - q_4 d\tau \). Provided that the conditions (15) hold, the following additional one is required to ensure that the considered periodic solution is LAS:

\[ \lambda_{13,4} = e^{P} \int_{0}^{P} D_{42} f_{42}(\bar{x}_2(\tau)) + D_{43} f_{43}(\bar{x}_3) - q_4 d\tau < 1 \iff \int_{0}^{P} D_{42} f_{42}(\bar{x}_2(\tau)) + D_{43} f_{43}(\bar{x}_3) - q_4 d\tau < 0 \]

\[ \iff \bar{x}_2^0 \left( 1 - e^{(\nu_1 - 1)q_2 P} \right) < a_{42} (e^{\nu_1 q_2 P} - 1) \]

Similarly to the previous case, (16) follows by studying the feasibility of the last inequality. □

References


