Inflation Derivatives and European Central Bank

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Abstract

European inflation and interest rates are closely related. In particular, we consider the relationship among European inflation, European Central Bank official interest rate and short-term interest rate in a stochastic continuous time setting for the valuation of inflation derivatives. We model the state variables in two different time scales and we provide some results about the existence and uniqueness of: (i) a strong solution of a more general jump-diffusion problem in which our model is included, and (ii) a solution, in the classical sense, of the valuation equation as a series of solutions of degenerate parabolic PDEs with non-local terms.

1 Introduction

The market of inflation-linked public debt has experienced a significant growth in euro area and in other major bond markets in recent years (De Cecco et al. [DCPP97], Favero et al. [FMP00]). This growth can be seen as a consequence of the credibility of central banks in delivering price stability in the respective countries. The credibility of central
banks and their clear mandate to preserve price stability has indeed helped to significantly
diminish uncertainty about future inflation and, consequently, demand for these instru-
ments does exist.
Indexation of debt is not a new idea. Deacon and Derry [DD98] note an early example
that occurs in 1742 when Massachusetts issued bills linked to the price of silver on London
Exchange. The issuance of sovereign bonds linked to euro area inflation began with the
introduction of bonds indexed to the French consumer price index (CPI) excluding tobacco
(Obligations Assimilables du Trésor indexées or OATis) in 1998. In 2003, Greece, Italy
and Germany have decided to issue inflation-linked bonds too. Despite that, the infla-
tion derivatives market is only on its infancy. Some typical examples are inflation caps which
pay out if the inflation exceeds a certain threshold over a given period, or inflation pro-
tected annuities that guarantee a real rate (given by the nominal return less the inflation
rate) of return at or above the inflation (for a detailed list of inflation derivatives, see
Hughston [HUG98]).
The pricing of inflation derivatives is related to both interest rate and foreign exchange
theory. It has been tackled, among others, by Barone and Castagna [BC97], Hugh-
ston [HUG98] and Jarrow and Yildirim [JY03], who proposed an approach based on
foreign currency and interest rate derivatives valuation. On the other hand, there ex-
ists some empirical and theoretical evidence that bond prices, inflation, interest rates,
monetary policy and output growth are related. Chacko and Das [CD02] introduce a gen-
eral framework for exponential-affine term structure models by incorporating some factors
which influence the marginal productivity of capital, and thus the interest rates, in the
economy. Piazzesi [PIA05] introduces a class of linear-quadratic jump-diffusion processes
in order to develop an arbitrage-free time-series model of yields in continuous time that
incorporates the central bank policy and estimates the model with the U.S. interest rates
and the Federal Reserve’s target.
The aim of this work is the valuation of inflation derivatives with payoff depending on
European inflation, on European Central Bank (henceforth ECB) official interest rate and
on the short-term interest rate at times between the present and the maturity date. For
this reason we investigate the relationship among these three quantities in a stochastic
continuous time setting. Distinct from previous research, we model the state variables in
two different time scales (deterministic and stochastic) and this hypothesis needs a slight
modification of the usual definition of the market price of risk. Another consequence is
the representation of the solution, in the classical sense, of the valuation equation as a
series of degenerate parabolic PDE with non-local terms. From the point of view of the
modelisation, the ECB interest rate evolves as a pure jump process with arrival intensities
that depend on decisions about the inflation rate and, consequently, on the official interest rate. The inflation rate is modeled as a step function and the short-term interest rate as a mean-reverting process.

The paper is organized as follows. In Section 2, we introduce the model under the historical probability measure. In Section 3 we specify the transfer to the risk neutral measures and Section 4 deals with the valuation equation.

2 The model

![Figure 1: Evolution of European Inflation, ECB and short-term interest rates in the period January 1999-February 2007 (daily data).](image)

The primary objective of the ECB is to maintain price stability, i.e., to keep inflation within a desired range (close to 2%). The inflation target is achieved through periodic adjustment to the ECB official interest rate target and, consequently, to short-term interest rates. The behavior just introduced is the core of this work. More in detail, we formulate a dynamical model under the historical probability measure that describes the existing relationship among inflation rate, EBC and short-term interest rate.

Figure 1 plots the ECB interest rate together with inflation rate and short-term interest rate. Looking at the figure, we can see two important facts about the official interest rate. First, the level of the rate is persistent, and its changes are often followed by additional changes in the same direction (especially in the last years).
Distinct from previous research, the evolution of the state variables is given in two different time scales, deterministic for the inflation, and stochastic for the interest rates. This is because the official announcements about the inflation rate are given every month whereas the changes of official interest rate occur unexpectedly (the period may largely vary). The modelisation of the ECB interest rate is based on the theory of Markov jump processes: we give two equivalent representations of the evolutions of this interest rate by imposing that the jump intensities are bounded functions and, consequently, by exploiting the technique of construct a Markov jump process with a bounded generator (Ethier and Kurtz [EK86]). Consistently with the empirical observation, we model the short-term interest rate as a mean-reverting process.

From now on, we consider $t \in [0,T]$ with $T < +\infty$, and we fix the space of probability $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the historical measure.

### 2.1 European Inflation

The value of European rate of inflation is officially made known once a month, hence we model it as a stochastic process that jumps at fixed times, with jump sizes depending on the previous value of the inflation and on the spread between the official ECB interest rate and the short-term rate, and it is constant between two jumps.

Specifically, using the usual convention that one year is an interval of length one, let $\mathcal{T} := \{t_i\}_{i=0,\ldots,M}$ ($t_i = \frac{i}{12}$) be the sequence of times at which the values of the inflation process, $\{\Pi_t\}_{t \in \mathcal{T}}$, are observed. The evolution is then given by

$$\begin{cases} 
\Pi_t = \Pi_{t_i}, & t_i \leq t < t_{i+1}, \\
\Pi_{t_{i+1}} = f\left(\Pi_{t_i}, R_{t_{i+1}}, R_{t_{i+1}}^{sh}\right) + \epsilon_{i+1}, & t = t_{i+1},
\end{cases}$$

(1)

where $f$ is a linear function defined by

$$f(\pi, r, z) = \alpha \pi + k^{\Pi}(\pi^* - \pi) + \beta (r - z),$$

(2)

with $\alpha$, $\beta \in \mathbb{R}$ and $k^{\Pi}$, $\pi^* \in \mathbb{R}_+$ constant parameters. The fluctuations $\{\epsilon_i\}_{i=0,\ldots,M}$ are i.i.d. random variables distributed according to the $N(0, v^2)$ law, and $\{R_t\}_{t \in [0,T]}$ and $\{R_{t_{i+1}}^{sh}\}_{t \in [0,T]}$ are the interest rate processes which will be introduced in Sections 2.2 and 2.3. From (2), it follows that the sample paths of the process $\{\Pi_t\}$ are step functions and that between two consecutive jumps, it satisfies the mean-reversion property, that is, it approaches to its long-period mean $\frac{k^{\pi}}{k^{\pi} - \alpha} \pi^*$ with speed $(k^{\pi} - \alpha)$. 
2.2 European Central Bank Interest Rate

The European Central Bank interest rate $R_t$ is modeled as a pure jump process with upward and downward jumps. It can be represented by the sum of two marked point processes (mpp) $(Y^U_n, T^U_n)_{n \geq 1}$ and $(Y^D_n, T^D_n)_{n \geq 1}$ (for the definition of a mpp see [BRE81, RUN03]). The sequences of random variables $(Y^U_n)_{n \geq 1}$ and $(Y^D_n)_{n \geq 1}$ are the upward and downward jump sizes, respectively. We suppose that the set of all possible jump sizes is finite and it does not contain the infinite, that is, $Y^U_n \in E^U = \{y_1, \ldots, y_m\}$, and $Y^D_n \in E^D = \{-y_1, \ldots, -y_m\}$ where $y_{\pm i} \in \mathbb{R}_+ \setminus \{0, \infty\}$ for every $i = 1, \ldots, m$ ($m < +\infty$). We adopt the notation $y_0 := 0$. The point processes $T^U_n$ and $T^D_n$ are the sequences of instants of occurrence of upward and downward jumps, respectively. We suppose that an upward jump and a downward jump cannot occur simultaneously, that is, $\{T^U_n\} \cap \{T^D_n\} = \emptyset$ for every $n \geq 1$. We define $\gamma^U(t) := Y^U_n$, for every $T^U_n \leq t < T^U_{n+1}$, namely, the piecewise constant, left-continuous time interpolation of the sequence $Y^U_n$. It follows that

$$\int_0^t \gamma^U(s) \, dN^U_s = \sum_{n=1}^{N^U_t} \gamma(T^U_n),$$

for every $t \geq 0$, where $N^U_t$ is the counting process of jumps upwards. Analogously for $\gamma^D(t)$. Therefore, we have

$$R_t = R_0 + \int_0^t (\gamma^U_s \, dN^U_s - \gamma^D_s \, dN^D_s),$$

where $R_0$ is a positive random variable. We suppose that the counting processes $N^U_t$ and $N^D_t$ have stochastic intensities $\lambda^U_t$ and $\lambda^D_t$.

Figure 1 shows that the sample path of the official interest rate is a step function and its steps are multiples of 25 basis points (bp). In continuous time, the ECB interest rate is a pure jump process given by (4) (where $\gamma^U(t) = \gamma^D(t) = 0.0025$ for every $t \geq 0$). Heuristically, the probability of an upward jump in $[t, t + dt]$, conditional on information up to time $t$, is given by $\lambda^U_t \, dt$, and the conditional probability to have a downward jump is $\lambda^D_t \, dt$. This means that ECB moves are of 0 bp, $y_{\pm 1}$ bp, $y_{\pm 2}$ bp, and so forth. We impose that, conditional on the occurrence of a jump, upwards or downwards, of the same size (in absolute value) the probability is given by $p_l$, for every $l = 1, \ldots, m$ ($\sum_{l=1}^m p_l = 1$). When the level of the official interest rate is low, there is the tendency to avoid further downward jumps, or, at least, to reduce the occurrence of this type of jumps. To reproduce this effect, we suppose that the intensities are functions of the current values of the inflation process and of the ECB interest rate $\lambda^U_t = \lambda^U(\pi, r)$ and $\lambda^D_t = \lambda^D(\pi, r)$ and bounded. Then, the quantity

$$\bar{\lambda} := \sup_{(\pi, r) \in \mathbb{R}^2} [\lambda^U(\pi, r) + \lambda^D(\pi, r)],$$

for every $t \geq 0$.
is well defined and a real parameter. Moreover, \( \lambda \) can be considered as the intensity of a new counting process \( N \) that encompasses the previous processes \( N^U \) and \( N^D \) ([EK86]): we use this fact to consider, henceforth, the ECB interest rate as the solution of the following stochastic equation driven by a Poisson process \( N_t \) with intensity \( \lambda \)

\[
R_t = R_0 + \int_0^t J \left( \Pi_{s^{-}}, R_{s^{-}}, \zeta_{N_{s^{-}}+1} \right) \ dN_s, \tag{6}
\]

where \( \{\zeta_n\}_{n \geq 0} \) are i.i.d. \((0,1)\)-uniform random variables, independent of \( N_t \),

\[
J(\pi, r, u) := \sum_{l=-m, m \neq 0}^m y_l \mathbf{1}_{(q_{-m-1}(\pi, r)+\ldots+q_{-1}(\pi, r),q_{-m-1}(\pi, r)+\ldots+q(\pi, r))(u)}, \tag{7}
\]

for \( u \in (0, 1) \), and

\[
q_{-l}(\pi, r) := \frac{pl \lambda^D(\pi, r)}{\lambda}, \quad q_l(\pi, r) := \frac{pl \lambda^U(\pi, r)}{\lambda}, \quad q_0(\pi, r) := 1 - \sum_{l=-m, m \neq 0}^m q_l(\pi, r) = 1 - \frac{\lambda^U(\pi, r)+\lambda^D(\pi, r)}{\lambda}, \tag{8}
\]

Note that the boundedness of the intensities \( \lambda^U_t \) and \( \lambda^D_t \) implies that \( R_t \in [0, r] \) for every finite \( R_0 \geq 0 \) and for every \( t \geq 0 \).

**Proposition 2.1** \((4) \) and \((6) \) are equivalent.

### 2.3 Short-term Interest Rate

We model the evolution of the short-term interest rate \( R^{sh} \) by the following mean-reverting equation

\[
\begin{cases}
        dR^{sh}_t = \ k^{sh} \left( b(R_t) - R^{sh}_t \right) \ dt + \sigma \left( |R_t - R^{sh}_t| \right) \sqrt{|R^{sh}_t|} \ dW_t, \\
        R^{sh}_0 > 0,
    \end{cases}
\]

where \( k^{sh} \in \mathbb{R}_+ \) is a constant parameter and \( \{W_t\}_{t \in [0,T]} \) is a standard Wiener process. The function \( b(r) \) is defined by

\[
b(r) = b_0 + b_1 r, \tag{10}
\]

where \( b_0, b_1 \in \mathbb{R} \) are constant parameters and the function \( \sigma(\cdot) \) is increasing and locally Hölder continuous with exponent \( \frac{1}{2} \). We introduce the dependence of short-term interest rate \( R^{sh}_t \) on \( R_t \) in the drift term and in the volatility term. In particular, we suppose that the function \( \sigma \) depends on the absolute value of the spread between the official and the short-term interest rates. Under some conditions on parameters and on the function \( \sigma \) (see Theorem A-1.1), we obtain the positivity of the process \( R_t \) for every \( t \geq 0 \) and for every initial value \( R^{sh}_0 \). Therefore, from now on, we omit the absolute value in the dynamics of \( R^{sh} \).
2.4 Well-posedness of the model

We suppose that the sources of randomness in (1), (6) and (9), that is \( \{ \epsilon_i \}_{i=0,\ldots,M}, \{ \zeta_n \}_{n\geq 0}, \{ N_t \}_{t\geq 0} \) and \( \{ W_t \}_{t\geq 0} \), are mutually independent. All information is given by the following filtration

\[ \mathcal{F}_t := \sigma \left( \left\{ \Pi_0, R_0, R_0^{sh}, \epsilon_{I(s)}, N_s, W_s, \zeta_{N_s}, s \leq t \right\} \right), \] (11)

where \( I(t) \) is the number of jumps of inflation up to time \( t \), namely,

\[ I(t) := \begin{cases} \max\{ i \geq 0 : t_i \geq t \}, & t \geq 0, \\ 0, & t < 0. \end{cases} \] (12)

The model introduced in Sections 2.1-2.3 is well posed, in the sense that there exists one and only one stochastic process \( X_t := (\Pi_t, R_t, R_t^{sh})' \) verifying (1), (6) and (9) in a strong sense, as we prove in the following theorem.

**Theorem 2.1** For every \( \Pi_0 \in \mathbb{R}, R_0 \in [0, r_0] \) and \( R_0^{sh} > 0 \), there exists one and only one stochastic process \( (\Pi_t, R_t, R_t^{sh})' \) defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \), \( \mathcal{F}_t \)-adapted, such that (1), (6) and (9) are \( \mathbb{P} \)-a.s. verified.

**Proof.** See Appendix. \( \square \)

3 Market Price of Risk

The market of every asset with price that depends on the processes described in Section 2 is incomplete, hence the principle of absence of arbitrage does not lead to a uniquely defined price. One actually obtains an entire range of prices ([BT85, EJ97]) and the preference structure of the investors has to come to play to determine the pricing measure. From the point of view of pricing, the problem then reduces to determine a specific martingale measure (the risk neutral measure) or, equivalently, a market price of risk. To do this, we apply Girsanov’s theorem for semimartingales ([JS87]) to characterize the infinitely many measures equivalent to the historical one, and among them, we choose those that preserve the structure of the model (1), (6) and (9). All details can be found in Appendix.

Consider the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) where \( \mathbb{P} \) is the historical measure and the filtration \( \mathcal{F}_t \) defined in (11). The state vector \( X_t = (\Pi_t, R_t, R_t^{sh})' \) is a special semimartingale ([PRO90]), hence there exists the canonical decomposition

\[ X_t = X_{0t} + M_t + A_t, \] (13)
where $X_0$ is the initial value, $M_i$ is a local martingale and $A_i$ is a predictable and finite variation process, for every $i = 1, 2, 3$. From (13), we derive the canonical decomposition of $X$ with respect to the infinitely many probability measures $\tilde{P}$, equivalent to $P$. Among all $\tilde{P}$, according a popular assumption in literature ([VAS77, CIR85]), we fix the measures under which $X$ evolves with the dynamics (1), (6) and (9) (with different parameters). We denote by $E$ the set of these measures. Applying the Girsanov’s theorem, it is possible to obtain the explicit expression of $\tilde{A}_i$, $i = 1, 2, 3$, the predictable and finite variation processes of the canonical decomposition of $X$ in $E$. Now, we want to derive the market price of risk; to do this, we need to modify its usual definition (see [RUN03] for an ‘usual’ definition) because the components of the state vector $X_t = (\Pi_t, R_t, R_t^{sh})'$ evolve in different (and disjoint, with probability 1) time scales. The new definition will be given using the processes $A$ and $\tilde{A}$.

We consider the time process that contains both kinds of jumps (deterministic for $\Pi_t$ and stochastic for $R_t$)

$$D_t = t + I_t,$$

for every $t \geq 0$, where $I_t$ is defined in (12). Then, we have

**Definition 3.1** The market price of risk is

$$\theta_t = \frac{dA_t}{dD_t} - \frac{d\tilde{A}_t}{dD_t},$$

where $A_t$ and $\tilde{A}_t$ are the predictable and finite variation processes of the canonical decomposition of $X$ under $P$ and $\tilde{P}$, respectively.

**Remark.** An explicit expression of the market price of risk can be obtained by specifying the form of the intensities $\lambda^U$, $\lambda^D$. For example, if we consider

$$\lambda^U(\pi, r) := c^U (\pi \land \overline{\pi} + \overline{\pi})_+ (\overline{\pi} - 0.005 - r)_+, \quad (\pi, r) \in \mathbb{R}^2,$$

$$\lambda^D(\pi, r) := c^D [\overline{\pi} - (\pi \lor (-\overline{\pi}))]_+ (r - 0.005)_+, \quad (\pi, r) \in \mathbb{R}^2,$$

where $c^U$, $c^D \in \mathbb{R}_+$ are parameters and $\overline{\pi}$, $\overline{\pi}$ are fixed large positive constants, the process $\theta_t$ is given by

$$\theta^U_{2t} - \theta^D_{2t} = \left[ (\alpha - \tilde{\alpha} + \overline{k} \Pi_t + (k \overline{k} - \overline{k}) \pi^* + (\beta - \tilde{\beta})(R_t - R_t^{sh}) \right] \mathbb{1}_{\mathcal{F}(t)}$$

$$\left[ k^{sh} (b(R_t) - R_t^{sh}) - \tilde{k}^{sh} (\tilde{b}(R_t) - R_t^{sh}) \right] \mathbb{1}_{\mathbb{R}_+ \setminus \mathcal{F}(t)},$$

where $\mathcal{F} = \{t_i\}_{i=0, \ldots, M}$ and

$$\theta^D_{2t} = \sum_{l=-m, \ m \neq 0}^{m} y_l (p_l c^D - \overline{p}_l c^D) [\overline{\pi} - (\Pi_t \lor (-\overline{\pi}))]_+ (R_t - 0.005)_+ \mathbb{1}_{\mathbb{R}_+ \setminus \mathcal{F}(t)},$$

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\[ \theta_{2t}^U = \sum_{l=-m, \ m \neq 0} y_l (p_l c^U - \tilde{p}_l c^U) (\Pi_t \wedge \pi + \pi)_+ (\gamma - 0.005 - R_t)_+ 1_{\mathbb{R}_+ - \pi}(t). \]

Note that, in this case, \( \theta_t \) is an affine function of the state vector \( X_t = (\Pi_t, R_t, R_{sh}^t)' \).

## 4 The Valuation Equation

In this section, we derive and solve the valuation equation that the discounted price \( P_t \) of a contingent claim with maturity \( T \) and payoff \( \Phi(T, X_t) \) satisfies if we specify a risk neutral measure \( \tilde{\mathbb{P}} \in \mathcal{E} \). The key idea is to divide the time interval \([0, T]\) into fixed subintervals delimited by inflation times \( \{t_i\}_{i=0, \ldots, M} \). In each subinterval, the price \( P_t \) satisfies a degenerate parabolic PDE with non-local term, hence \( P_t \) can be considered as a series of solutions, in the classical sense, of these integro-differential equations (the integral term is due to the presence of jumps of ECB interest rate). We consider the market price of risk defined in (3.1).

Suppose that \( T = t_M \) (i.e., the last jump time of inflation). Under no arbitrage assumption, the price \( P_t \) of a contingent claim can be expressed as the discounted expected payoff with respect to the risk neutral measure \( \tilde{\mathbb{P}} \), namely

\[ P_t = \exp \left( \int_{t_i}^t R_{sh}^s ds \right) M_t. \]  

It is worth remembering that \( M_t \) is a local martingale with respect to \( \tilde{\mathbb{P}} \) ([HP81]). We look for \( P_t \) in the form

\[ P_t := \varphi(t, X_t) \]  

with

\[ \varphi(t, x) = \varphi^i(t, x), \quad t_i \leq t < t_{i+1}, \ i = 0, \ldots, M - 1, \]

\[ \varphi(T, x) = \Phi(T, x), \]  

and \( \varphi^i \) sufficiently smooth, for every \( i = 0, \ldots, M - 1 \). Now, we write down the valuation equation that each \( \varphi^i \) satisfies. We have

**Theorem 4.1** Given \( \varphi \) as in (21), where the functions \( \varphi^i \) are sufficiently smooth, then (19)-(20) hold if and only if the functions \( \varphi^i \) satisfy

\[ \frac{\partial \varphi^i}{\partial t}(t, \pi, r, z) + k_{sh} (b(r) - z) \frac{\partial \varphi^i}{\partial z}(t, \pi, r, z) + \frac{1}{2} \sigma^2 (|r - z|) z \frac{\partial^2 \varphi^i}{\partial z^2}(t, \pi, r, z) + J^D(t, \pi, r, z) + J^U(t, \pi, r, z) - \varphi^i(t, \pi, r, z)z = 0, \]

\[ t \in [t_i, t_{i+1}), \ i = 0, \ldots, M - 1, \ \pi \in \mathbb{R}, \ r \in [0, \pi], \ z > 0, \]  

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\( \varphi^i(t_{i+1}, \pi, r, z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi^{i+1}(t_{i+1}, f(\pi, r, z) + u) \exp \left( -\frac{u^2}{2\sigma^2} \right) \, du, \quad i = 0, 1, \ldots, M - 2, \)

\( \varphi^{M-1}(t_M, \pi, r, z) = \Phi(t_M, \pi, r, z), \quad \pi \in \mathbb{R}, \ r \in [0, \bar{r}], \ z > 0, \) \hfill (24)

where

\[
J^D(t, \pi, r, z) := \sum_{l=-m}^{m} \left[ \varphi^i(t, \pi, r - y_l, z) - \varphi^i(t, \pi, r, z) \right] p_l \lambda^D(\pi, r), \hfill (25)
\]

and

\[
J^U(t, \pi, r, z) := \sum_{l=-m}^{m} \left[ \varphi^i(t, \pi, r + y_l, z) - \varphi^i(t, \pi, r, z) \right] p_l \lambda^U(\pi, r). \hfill (26)
\]

**Proof.** See Appendix. \( \square \)

The sequence of problems (22)-(24) can be solved recursively given that one can solve, for every fixed \( \pi \in \mathbb{R}, \)

\[
\begin{cases}
\frac{\partial \varphi^i}{\partial t}(t, \pi, r, z) + \kappa^h \left( b(r) - z \right) \frac{\partial \varphi^i}{\partial z}(t, \pi, r, z) + \frac{1}{2} \sigma^2 \left( |r - z| \right) \frac{\partial^2 \varphi^i}{\partial z^2}(t, \pi, r, z) + J^D(t, \pi, r, z) + J^U(t, \pi, r, z) - \varphi^i(t, \pi, r, z) = 0, & t \in [t_i, t_{i+1}), \ r \in [0, \bar{r}], \ z > 0, \\
\varphi^i(t_{i+1}, \pi, r, z) = \Phi^i(\pi, r, z), & r \in [0, \bar{r}], \ z > 0.
\end{cases} \hfill (27)
\]

Without loss of generality, we can take \( i = 0 \) in (27) and thereby obtain a parabolic PDE with non-local terms and terminal condition. This problem is degenerate in various ways, therefore standard results do not applied. Under some conditions on payoff function, the existence and uniqueness of a solution, in the classical sense, of the valuation problem is obtained as we prove in the following theorem.

**Theorem 4.2** Suppose \( \Phi \) is bounded and locally Lipschitz continuous on \([0, \bar{r}] \times (0, \infty)\). Then (22)-(24) has one and only one solution \( \varphi^i \), for every \( i = 0, \ldots, M \), in

\[
S = \{ f \in C([0, t_1] \times [0, \bar{r}] \times (0, \infty)) : \forall r \in [0, \bar{r}], \ f(\cdot, r, \cdot) \in C^{1,1}([0, t_1] \times (0, \infty)) \}. \hfill (28)
\]

**Proof.** We prove the existence first.

Using the operators

\[
Af(r, z) := k^h \left( b(r) - z \right) \frac{\partial f}{\partial z}(r, z) + \frac{1}{2} \sigma^2 \left( |r - z| \right) \frac{\partial^2 f}{\partial z^2}(r, z), \hfill (29)
\]

\[
Bf(r, z) := J^D(t, \pi, r, z) + J^U(t, \pi, r, z) - f(t, \pi, r, z), \hfill (30)
\]
We are going to prove that (34) has a unique solution and consider the following terminal-boundary value problem

$$
\begin{cases}
\frac{\partial \varphi}{\partial t}(t, r, z) + (A + B)\psi(t, r, z) = 0, & 0 \leq t < t_1, \ z > 0 \\
\psi(t_1, r, z) = \Phi(r, z).
\end{cases}
$$

Let

$$
\varphi(t, r, z) := \mathbb{E}_{t, r, z} \left[ \Phi \left( R_{t_1}, R_{t_1}^{sh} \right) e^{-\int_{t_1}^{t} R_{s}^{sh} ds} \right],
$$

where \( (R_t, R_t^{sh}) \) is the solution of

$$
\begin{cases}
dR_t = J(\pi, R_{t-}, \zeta_{N_{t-}}) dN_t, \\
dR_t^{sh} = k^{sh} \left( b(R_t) - R_t^{sh} \right) dt + \sigma \left( |R_t - R_t^{sh}| \right) \sqrt{R_t^{sh}} dW_t.
\end{cases}
$$

For \( 0 < l < L < \infty \), consider the following terminal-boundary value problem

$$
\begin{cases}
\frac{\partial \varphi}{\partial t}(t, r, z) + (A + B)\psi(t, r, z) = 0, & 0 \leq t < t_1, \ z \in (l, L), \\
\psi(t, r, z) = \varphi(t, r, z), & t = t_1 \text{ or } z = l, L.
\end{cases}
$$

We are going to prove that (34) has a unique solution \( \psi \in C([0, t_1) \times (l, L)) \).

Let \( A^r \) be the operator obtained by viewing \( r \) as a parameter in (29), namely

$$
A^r f(z) = k^{sh} \left( b(r) - z \right) f'(z) + \frac{1}{2} \sigma^2 \left( |r - z| \right) z f''(z),
$$

and consider the following terminal-boundary value problem

$$
\begin{cases}
\frac{\partial \varphi^r}{\partial t}(t, z) + A^r \varphi^r(t, z) = -B \varphi(t, r, z), & 0 \leq t < t_1, \ z \in (l, L), \\
\psi(t, r, z) = \varphi^r(t, r, z), & t = t_1 \text{ or } z = l, L.
\end{cases}
$$

By Lemma A-1.6 in Appendix, \( B \varphi \) is Hölder continuous in \( t \), uniformly with respect to \( (r, z) \in [0, \bar{r}] \times [l, L] \). Therefore, by Lemma A-1.7 in Appendix, for each \( r \in [0, \bar{r}] \), (36) has one and only one solution \( \psi^r \in C^{1,2}([0, t_1) \times (l, L)) \cap C([0, t_1] \times (l, L)) \). In addition, \( \psi(t, r, z) := \psi^r(t, z) \) admits the stochastic representation

$$
\psi(t, r, z) = \mathbb{E}_{t, z} \left[ \varphi \left( t_1 \land \gamma^r(t), r, R_{t_1 \land \gamma^r(t)}^{sh, r} \right) \right] - \mathbb{E}_{t, z} \left[ \int_{t}^{t_1 \land \gamma^r(t)} B \varphi \left( s, r, R_{s}^{sh, r} \right) ds \right],
$$

where \( R_{t_1}^{sh, r} \) is the solution of

$$
dR_{t_1}^{sh, r} = k^{sh} \left( b(r) - R_{t_1}^{sh, r} \right) dt + \sigma \left( |r - R_{t_1}^{sh, r}| \right) \sqrt{R_{t_1}^{sh, r}} dW_t,
$$

and

$$
\gamma^r(t) := \inf \left\{ s \geq t : R_{s}^{sh, r} \notin (l, L) \right\}.
$$

Therefore \( \psi \) belongs to \( C([0, t_1] \times [0, \bar{r}] \times [l, L]) \). Now considering the process \( (R_t, R_t^{sh}) \) defined in (33) and

$$
\gamma(t) := \inf \left\{ s \geq t : R_{s}^{sh} \notin (l, L) \right\}.
$$
By Itô's formula we have
\[
\psi(t, r, z) = \mathbb{E}_{t, r, z} \left[ \varphi \left( t_1 \cdot \gamma(t), R_{t_1 \cdot \gamma(t)}, R_{t_1 \cdot \gamma(t)}^{sh} \right) e^{-\int_{t_1}^{t_1 \cdot \gamma(t)} R_u^{sh} du} \right] + \mathbb{E}_{t, r, z} \left[ \int_{t_1}^{t_1 \cdot \gamma(t)} B(\varphi - \psi)(s, R_s, R_s^{sh}) e^{-\int_{t_1}^{t_1 \cdot \gamma(t)} R_u^{sh} du} ds \right],
\]

hence, from (32),
\[
\psi(t, r, z) = \mathbb{E}_{t, r, z} \left[ \Phi \left( R_{t_1}, R_{t_1}^{sh} \right) e^{-\int_{t_1}^{t_1 \cdot \gamma(t)} R_u^{sh} du} \right] + \mathbb{E}_{t, r, z} \left[ \int_{t_1}^{t_1 \cdot \gamma(t)} B(\varphi - \psi)(s, R_s, R_s^{sh}) e^{-\int_{t_1}^{t_1 \cdot \gamma(t)} R_u^{sh} du} ds \right] = \varphi(t, r, z) + \mathbb{E}_{t, r, z} \left[ \int_{t_1}^{t_1 \cdot \gamma(t)} B(\varphi - \psi)(s, R_s, R_s^{sh}) e^{-\int_{t_1}^{t_1 \cdot \gamma(t)} R_u^{sh} du} ds \right].
\]

Therefore, for \(0 \leq t \leq t_1\),
\[
\sup_{[0, \gamma(t) \times [l, L], \gamma(t)}} \left| (\psi - \varphi)(t, r, z) \right| \leq c \sup_{[t]} \left| (\psi - \varphi)(s, r, z) \right| ds,
\]
for some constant \(c > 0\). Then, by Gronwall’s lemma,
\[
\psi(t, r, z) = \varphi(t, r, z), \quad \forall t \in [0, t_1], \quad r \in [0, \overline{r}], \quad z \in [l, L],
\]
and \(\varphi\) satisfies (34). Since \((l, L)\) was arbitrary, this proves that \(\varphi\) belongs to \(S\) and satisfies (31), i.e. (27).

Next, we turn to uniqueness of the solution to (27). Let \(\bar{\varphi} \in S\) be another solution of (27). By applying Itô’s formula to \(\bar{\varphi}\) and taking into account Theorem A-1.1, we obtain
\[
\bar{\varphi}(t, r, z) = \mathbb{E}_{t, r, z} \left[ \Phi \left( R_{t_1}, R_{t_1}^{sh} \right) e^{-\int_{t_1}^{t_1 \cdot \gamma(t)} R_u^{sh} du} \right] = \varphi(t, r, z).
\]

\[\square\]

**Remark.** By the same arguments used in Theorem 4.2, we can see that the terminal-boundary value problem
\[
\begin{cases}
\frac{\partial \varphi}{\partial t}(t, r, z) + (A + B)\varphi(t, r, z) = 0, & 0 \leq t < t_1, \quad z \in \left(\frac{1}{n}, n\right), \\
\varphi(t, r, z) = \Phi(r, z), & t = t_1 \text{ or } z = \frac{1}{n}, n,
\end{cases}
\]
has a unique solution \(\varphi \in C \left([0, t_1] \times [0, \overline{r}] \times (\left[\frac{1}{n}, n\right])\right), \varphi(\cdot, r, \cdot) \in C^{1,2} \left([0, t_1] \times (\left[\frac{1}{n}, n\right])\right).

Then, by Itô’s formula,
\[
\varphi^n(t, r, z) = \mathbb{E}_{t, r, z} \left[ \Phi \left( R_{t_1 \cdot \gamma^n(t)}, R_{t_1 \cdot \gamma^n(t)}^{sh} \right) e^{-\int_{t_1}^{t_1 \cdot \gamma^n(t)} R_u^{sh} du} \right],
\]
where \((R_t, R_t^{sh})\) is defined by (33) and
\[
\gamma^n(t) = \inf \left\{ s \geq t : R_s^{sh} \notin \left(\frac{1}{n}, n\right) \right\}.
\]

Therefore
\[
|\varphi^n(t, r, z) - \varphi(t, r, z)| \leq 2 \sup_{[0, \overline{r}] \times (0, \infty)} \left| \Phi(r, z) \right| \mathbb{P}_{t, r, z}(\gamma^n(t) \leq t_1),
\]
and the right hand side converges to zero as \(n\) tends to infinity, uniformly for \(t \in [0, t_1], \quad r \in [0, \overline{r}]\) and \(z\) varying in a compact subset of \((0, \infty)\) by Theorem A-1.1.
Theorem A-1.1 The function $\sigma$ in (9) is increasing and locally Hölder continuous with exponent $\frac{1}{2}$, and the conditions

\[ 0 < \sigma_0 \leq \sigma(q) \leq \sigma_1 \sqrt{1 + q}, \quad \sigma_0, \sigma_1, q \in \mathbb{R}_+, \tag{39} \]

\[ \frac{1}{2} \sigma_1^2 (1 + \tau) < k^{sh} b_0, \quad \sigma_1, \tau, k^{sh} \in \mathbb{R}_+ (\tau > 0), \quad b_0 \in \mathbb{R} \tag{40} \]

hold. Then, for every $R_0^{sh} > 0$, the process $R_t^{sh}$ satisfies $\mathbb{P}$-a.s.

\[ R_t^{sh} \geq 0, \quad \text{for every } t \geq 0. \]

Proof. For $\phi \in C^2_b(\mathbb{R})$, we define

\[ A\phi(r, z) := k^{sh}(b_0 + b_1 r - z)\phi'(z) + \frac{1}{2} \sigma^2(|r - z|)z\phi''(z). \tag{41} \]

Given $\alpha > 0$, consider $\phi^n \in C^2_b(\mathbb{R})$ such that $\phi^n(z) > 0$ and

\[ \phi^n(z) := z^{-\alpha}, \quad z \geq \frac{1}{n}. \tag{42} \]

Then, for $z \geq \frac{1}{n}$,

\[ A\phi^n(r, z) := \alpha z^{-(\alpha + 1)} \left\{ -k^{sh}(b_0 + b_1 r - z) + \frac{1}{2} \sigma^2(|r - z|)(\alpha + 1) \right\}. \tag{43} \]

Recall that $\sigma$ satisfies (40) and choose $\alpha > 0$ such that

\[ -k^{sh} b_0 + \frac{1}{2} \sigma_1^2 (1 + \tau)(\alpha + 1) < 0. \tag{44} \]

Let $\delta$ be such that

\[ \max_{0 \leq z \leq \delta} \left\{ -k^{sh} b_0 + \frac{1}{2} \sigma_1^2 (1 + \tau)(\alpha + 1) + \left( k^{sh} + \frac{1}{2} \sigma_1^2 (\alpha + 1) \right) z \right\} \leq 0. \tag{45} \]

Then, for $n > \frac{1}{\delta}$,

\[ \phi^n(z) \leq 0, \quad \text{for every } z \in [0, \tau]. \tag{46} \]

Let $z_{\delta}$ be such that $z_{\delta} < R_0^{sh}$ and $z_{\delta} < \delta$. For $n > \frac{1}{z_{\delta}}$, set

\[ \gamma^n := \inf\{t \geq 0 : R_t^{sh} \leq \frac{1}{n}\}, \tag{47} \]
and

\[\gamma_1 := \inf\{t \geq 0 : R^s_{t} \leq z_{\delta}\}, \quad (\gamma_1 := +\infty, \text{ if } R^s_{t} > z_{\delta}, \text{ for every } t \geq 0),\]

\[\sigma_1 := \inf\{t \geq \gamma_1 : R^s_{t} \geq \delta\},\]

\[\gamma_2 := \inf\{t \geq \sigma_1 : R^s_{t} \leq z_{\delta}\},\]

and so on. Note that \(\gamma_1 < \gamma^n \wedge \sigma_1\) and that, for all \(s \in [\gamma_1, \gamma^n \wedge \sigma_1]\), we have that \(R^s_{t} \in [\frac{1}{n}, \delta]\). Let \(E\) be the expectation starting at \(z_0 = R^s_{0}\) (analogously for \(P\)). By Itô’s formula and (46), we have

\[
E\left[\phi^n\left(R^s_{t \wedge \gamma^n}\right) - \phi^n\left(R^s_{t \wedge \gamma^n}\right)\right] = E\left[\int_{t \wedge \gamma^n} A\phi^n\left(R^s_{s}\right) \, ds\right] \leq 0,
\]

for any \(z_0 \geq z_{\delta}\). Then

\[
E\left[\mathbb{I}_{\{\gamma^n \leq t \wedge \sigma_1\}} \phi^n\left(R^s_{\gamma^n}\right)\right] \leq z_{\delta}^{-\alpha},
\]

and \(E\left[\mathbb{I}_{\{\gamma^n \leq t \wedge \sigma_1\}} \phi^n\left(R^s_{\gamma^n}\right)\right] = n^{\alpha} \mathbb{P}\{\gamma^n \leq t \wedge \sigma_1\}\), hence \(\mathbb{P}\{\gamma^n \leq t \wedge \sigma_1\} \leq z_{\delta}^{-\alpha}n^{-\alpha}\). By the strong Markov property, it follows

\[
\mathbb{P}\{\gamma^n \leq \sigma_k \wedge t | \gamma^n > \gamma_k\} = \mathbb{P}_{z_{\delta}}\{\gamma^n \leq t \wedge \sigma_1\} \leq z_{\delta}^{-\alpha}n^{-\alpha},
\]

for every \(k \geq 2\). Since \(\{\sigma_k\}_{k \geq 1}\) converges to \(+\infty\) as \(k \rightarrow +\infty\) with probability \(\mathbb{P} = 1\), one gets

\[
\mathbb{P}\{\gamma^n \leq t\} = \mathbb{P}\{\gamma^n \leq \sigma_1 \wedge t\} + \sum_{k=1}^{\infty} \mathbb{P}\{\sigma_k < \gamma^n \leq t \wedge \sigma_k+1\}.
\]

Now,

\[
\sum_{k=k_0}^{\infty} \mathbb{P}\{\sigma_k < \gamma^n \leq t \wedge \sigma_k+1\} \leq \mathbb{P}\{\sigma_{k_0} \leq t\} \rightarrow 0, \quad \text{as } k_0 \rightarrow +\infty,
\]

because the events \(\{\sigma_k < \gamma^n \leq t \wedge \sigma_k+1\}\) are disjoint. On the other hand,

\[
\sum_{k=1}^{k_0-1} \mathbb{P}\{\sigma_k \leq \gamma^n \leq t \wedge \sigma_k+1\} - \sum_{k=1}^{k_0-1} \mathbb{P}\{\gamma_k+1 < \gamma^n \leq t \wedge \sigma_k+1\} =
\]

\[
\leq (k_0 - 1)z_{\delta}^{-\alpha}n^{-\alpha}.
\]

Summing up, we have

\[
\mathbb{P}\{\gamma^n \leq t\} \leq k_0z_{\delta}^{-\alpha}n^{-\alpha} + \mathbb{P}\{\sigma_{k_0} \leq t\},
\]

and

\[
\limsup_{n \rightarrow +\infty} \mathbb{P}\{\gamma^n \leq t\} \leq 0.
\]

Since \(\gamma^n > 0\), from (47) it follows that \(R^s_{t} \geq 0\) for every \(t \geq 0\).

**Proof of Theorem 2.1.** The proof relies on the following lemma.
Lemma A-1.1 For any finite \{\mathcal{F}_t\}-stopping times \tau ad for every pair of \mathcal{F}_\tau-measurable random variable \((\rho_0, Z_0), Z_0 \in \mathbb{R}\)-valued, \rho_0 with values in a countable subset of \mathbb{R}, there exists one and only one stochastic process on \((\Omega, \mathcal{F}, \mathbb{P}), Z_t^{\rho_0, Z_0}\), that is \{\mathcal{F}_{\tau+t}\}-adapted (i.e. for each \(t \geq 0\), \(Z_t^{\rho_0, Z_0}\) is \(\mathcal{F}_{\tau+t}\)-measurable) and satisfies, \(\mathbb{P}\)-almost surely,
\[
Z_t = Z_0 + \int_t^{\tau+t} k^{sh} (b(\rho_0) - Z_{s-\tau}) \, ds + \int_t^{\tau+t} \sigma (|\rho_0 - Z_{s-\tau}|) \sqrt{|Z_{s-\tau}|} \, dW_s. \tag{57}
\]

Proof of Lemma A-1.1. Let \(\mathcal{F}_\tau^t := \mathcal{F}_{\tau+t}\), \(W_t^\tau := W_{\tau+t} - W_t\). Then \(W_t^\tau\) is an \(\{\mathcal{F}_t^\tau\}\)-standard Wiener process and (57) is equivalent to
\[
Z_t = Z_0 + \int_0^t k^{sh} (b(\rho_0) - Z_s) \, ds + \int_0^t \sigma (|\rho_0 - Z_s|) \sqrt{|Z_s|} \, dW_s^\tau. \tag{58}
\]
For every fixed value \(r\) of \(\rho_0\), consider
\[
Z_t^r = Z_0 + \int_0^t k^{sh} (b(r) - Z_s) \, ds + \int_0^t \sigma (|r - Z_s|) \sqrt{|Z_s|} \, dW_s^\tau. \tag{59}
\]
As mentioned in [IW81] Theorem 3.2 and the subsequent Corollary can be localized, hence applied to the coefficients of (59) as well. Therefore there exists a function \(F\) such that, for every \(t \geq 0\), \(Z_t^r := F(r, Z_0, W^\tau)_t\) is an \(\{\mathcal{F}_t^\tau\}\)-measurable random variable and the process \(Z^r\) satisfies (59) \(\mathbb{P}\)-a.s.. Since \(\rho_0\) takes values in a countable set, \(Z_t^{\rho_0, Z_0} := F(\rho_0, Z_0, W^\tau)_t\) is still \(\{\mathcal{F}_t^\tau\}\)-measurable and the process \(Z_t^{\rho_0, Z_0}\) satisfies (58), \(\mathbb{P}\)-a.s.. \(\Box\)

Now, let us turn on the proof of the theorem 2.1. First of all, notice that \(\{\tau_n\}_n \cap \{t_i\}_i = \emptyset\), \(\mathbb{P}\)-a.s.. Clearly, it is enough to show that, given \((\Pi_{t_i}, R_{t_i}, R^{sh}_{t_i})_i\), one can construct \((R_t, R^{sh}_t)\) in \([t_i, t_{i+1}]\), \(i = 0, \ldots, M - 1\). Let
\[
\tau_n^i := \inf \{t \geq t_i : N_t - N_{t_i} \geq n\}, \quad n \in \mathbb{Z}_+.
\]
Assuming inductively that \((R_{\tau_n^i \wedge t_{i+1}}, R_{\tau_n^i \wedge t_{i+1}}^{sh})\) is given and \(\mathcal{F}_{\tau_n^i \wedge t_{i+1}}\)-measurable and define
\[
R_{\tau_n^i \wedge t_{i+1}}^{sh} := Z_s^{(\tau_{n+1}^i \wedge t_{i+1}), R_{\tau_n^i \wedge t_{i+1}}, R_{\tau_n^i \wedge t_{i+1}}^{sh}}, \quad 0 \leq s \leq \tau_n^i \wedge t_{i+1} - \tau_{n+1}^i \wedge t_{i+1},
\]
for \(0 \leq s \leq (\tau_n^i \wedge t_{i+1}) - (\tau_{n+1}^i \wedge t_{i+1})\), where the right hand side is the process defined in Lemma A-1.1, and for \(0 \leq s < (\tau_{n+1}^i \wedge t_{i+1} - (\tau_n^i \wedge t_{i+1})\),
\[
R_{(\tau_n^i \wedge t_{i+1})+s} = R_{\tau_n^i \wedge t_{i+1}},
\]
\[
R_{(\tau_{n+1}^i \wedge t_{i+1})+s} = R_{(\tau_{n+1}^i \wedge t_{i+1})} + J \left( \Pi_t, R_{(\tau_{n+1}^i \wedge t_{i+1})}, \zeta N_{(\tau_{n+1}^i \wedge t_{i+1})} - N_{(\tau_{n+1}^i \wedge t_{i+1})} \right).
\]
Then, by induction on \(n\), \((R_t, R^{sh}_t)\) is defined for \(t \in [t_i, t_{i+1}]\), is \(\mathcal{F}_t\)-adapted and satisfies (1), (6) and (9) for \(t \in [t_i, t_{i+1}]\). \(\Box\)
Now, we derive the canonical decomposition of each component of \( X = (\Pi, R, \Rsh) \)' with respect to \( \mathbb{P} \). Let’s start from the process \( \Pi_t \) defining the càdlàg and \( \mathcal{F}_t \)-adapted process

\[
a_t := (\alpha - k\Pi - 1)\Pi_t + k\Pi^{\pi^*} + \beta \left( R_t - \Rsh_t \right) .
\]  

(60)

We will denote by \( \mathcal{B}(\mathbb{R}^k) \) the Borel \( \sigma \)-algebra of \( \mathbb{R}^k \), for all \( k \geq 1 \). The measure associated with the jumps of \( \Pi_t \) is

\[
\mu^\pi ([0, t] \times B) := \int_{0^+}^{t^+} \mathbf{1}_B(\epsilon_{I_s} + a_{s-})dI_s, \quad B \in \mathcal{B}(\mathbb{R}),
\]  

(61)

its compensator is

\[
\nu^\pi ([0, t] \times B) := \int_{0^+}^{t^+} \mathcal{N}(B, a_{s-})dI_s, \quad B \in \mathcal{B}(\mathbb{R}),
\]  

(62)

where for all \( B \in \mathcal{B}(\mathbb{R}) \) and for all \( a \in \mathbb{R} \),

\[
\mathcal{N}(B, a) = \int_B \frac{1}{\sqrt{2\pi v}} \exp \left\{ -\frac{(\pi - a)^2}{2v^2} \right\} d\pi,
\]  

(63)

and the process \( I_t \) is defined by

\[
I(t) := \begin{cases} 
\max\{i \geq 0 : t_i \geq t\}, & t \geq 0, \\
0, & t < 0.
\end{cases}
\]  

(64)

**Lemma A-1.2** The canonical decomposition of the process \( \Pi_t \) is

\[
\Pi_t = \Pi_0 + M_t^\Pi + A_t^\Pi,
\]  

(65)

where

\[
M_t^\Pi = \int_{[0,t] \times \mathbb{R}} \pi d(\mu^\pi - \nu^\pi) = \int_{0^+}^{t^+} \epsilon_{I_s}dI_s,
\]  

(66)

is the local martingale and

\[
A_t^\Pi = \int_{[0,t] \times \mathbb{R}} \pi d\nu^\pi = \int_{0^+}^{t^+} a_{s-}dI_s,
\]  

(67)

is the predictable and finite variation process.

**Proof.** The process \( \Pi_t \) is a pure jump semimartingale and \( \mu^\pi \) is the measure associated with its jumps. Therefore the assertion follows immediately from the fact that \( \nu^\pi \) is the compensator of \( \mu^\pi \).

\( \square \)

The measure associated to the jumps of \( R_t \) is concentrated in \( E := \{y_1, \ldots, y_m\} \cup \{y_{-1}, \ldots, y_{-m}\} \cup \{0\} = E^U \cup E^D \cup \{0\} \) (see Section 2.2) and given by

\[
\mu^R ([0, t] \times E \setminus \{0\}) = \int_{0}^{t} \mathbf{1}_{E \setminus \{0\}} \left( J \left( \Pi_{s-}, R_{s-}, \zeta_{N_{s-} + 1} \right) dN_s \right).
\]  

(68)
Its compensator is
\[ \nu^R ([0, t] \times E^D) = \int_0^t p_t \lambda^U (\Pi_s, R_s) \, ds, \]  
(69)  
\[ \nu^R ([0, t] \times E^U) = \int_0^t p_t \lambda^D (\Pi_s, R_s) \, ds, \]  
(70)
where \( l = -m, \ldots, -1, 1, \ldots, m \ (m < \infty). \)

**Lemma A-1.3** The canonical decomposition of the process \( R_t \) is
\[ R_t = R_0 + M_t^R + A_t^R, \]  
(71)
where
\[ M_t^R = \int_{[0,t] \times \mathbb{R}} r \, d (\mu^R - \nu^R), \]  
(72)
is the local martingale and
\[ A_t^R = \int_{[0,t] \times \mathbb{R}} \sum_{l=-m, \ m \neq 0} \lambda_U (\Pi_s, R_s) - \lambda_D (\Pi_s, R_s) \, ds, \]  
(73)
is the predictable and finite variation process.

**Proof.** Analogously to \( \Pi_t \). \( \square \)

**Lemma A-1.4** The canonical decomposition of the process \( R^{sh}_t \) is
\[ R_t^{sh} = R_0^{sh} + M_t^{R^{sh}} + A_t^{R^{sh}}, \]  
(74)
where
\[ M_t^{R^{sh}} = \int_0^t \sigma (|R_s - R_s^{sh}|) \sqrt{R_s^{sh}} \, dW_s, \]  
(75)
is the local martingale and
\[ A_t^{R^{sh}} = \int_0^t k^{sh} (b(R_s) - R_s^{sh}) \, ds, \]  
(76)
is the predictable and finite variation process.

**Proof.** The assertion follows from the integral form of (9). \( \square \)

Using the previous decompositions of the processes, we give an explicit “good” version ([JS87], Chapter II, Proposition 2.9) of the characteristics of \( X \). We denote by \( \delta_x \) the Dirac’s delta centered at \( x \) and we define
\[ D_t := t + I_t. \]  
(77)
Note that
\[ dI_t = \mathbb{1}_\tau(t) dD_t, \]  
and
\[ dt = \mathbb{1}_{\mathbb{R}_+ - \tau(t)} dD_t, \]  
where \( I_t \) is defined in (64) and \( \tau = \{t_i\}_{i=0,\ldots,M} \) is the sequence of jump times of inflation.

**Lemma A-1.5** A good version of the characteristics of \( X_t = (\Pi_t, R_t, R_t^{sh})' \) under \( \mathbb{P} \) is given by the triple \((A, C, \nu)\):

\[
A = \begin{pmatrix}
\int_{[0,t] \times \mathbb{R}} \pi \ d\nu \\
\int_{[0,t] \times \mathbb{R}} r \ d\nu \\
\int_0^t k^{sh} (b(R_s) - R_s^{sh}) \ ds
\end{pmatrix}
= \begin{pmatrix}
A_t^\Pi \\
A_t^R \\
A_t^{R^{sh}}
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \int_0^t \sigma^2 (|R_s - R_s^{sh}|) R_s^{sh} ds
\end{pmatrix},
\]

\[
\nu(dt, d\pi, dr, dz) := \left\{ N(d\pi, a_{\tau-}) \times \delta_0(dr) \times \delta_0(dz) \mathbb{1}_\tau(t) + \right.
\]
\[\left. \delta_0(d\pi) \times \sum_{l=-m, m \neq 0} p_l (\delta_{-l}(dr) \lambda^D(\Pi_t, R_t) + \delta_l(dr) \lambda^U(\Pi_t, R_t)) \times \delta_0(dz) \mathbb{1}_{\mathbb{R}_+ - \tau(t)} \right\} dD_t,
\]

where \( A_t^\Pi, A_t^R \) and \( A_t^{R^{sh}} \) are given by (67), (73) and (76) respectively. \( N(d\pi, a) \) is defined by (63), \( \lambda^U_t \) and \( \lambda^D_t \) are the stochastic intensities of \( N_t^U \) and \( N_t^D \), respectively, \( b \) is defined in (10) and \( \sigma \) in (39) and (40).

**Proof.** Since the first two components of \( X, \Pi \) and \( R \), have no common jumps with probability \( \mathbb{P} = 1 \), and the third component, \( R_t^{sh} \), has no jumps, the compensator of the measure associated with the jumps of \( X \) is determined by the compensator of the jump measures of \( \Pi \) and \( R \), which, in turn, are given by (62), (69) and (70).

The components of the process \( A \) are the processes \( A_t^\Pi, A_t^R \) and \( A_t^{R^{sh}} \) given by (67), (73) and (76) respectively.

Finally, the matrix \( C \) is defined by ([JS87], Chapter II, Definition 2.6)

\[
C_{(ij)t} = \langle X^c_{it}, X^c_{jt} \rangle_t,
\]

where \( X^c \) is the continuous martingale part of \( X \). In particular we have

\[
X^c_{1t} = X^c_{2t} = 0, \quad X^c_{3t} = M^{R^{sh}}_t = \int_0^t \sigma \left( |R_s - R_s^{sh}| \right) \sqrt{R_s^{sh}} dW_s.
\]

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Therefore, the only non zero element of $C_t$ is $C_{(33)h}$ which equals $\int_0^t \sigma^2 (|R_s - R^{sh}_s|) R^{sh}_s ds$. □

Now, let $\tilde{P}$ be a probability measure equivalent to $P$ and $\left( \tilde{A}, \tilde{C}, \tilde{\nu} \right)$ be the characteristics of $X$ with respect to $\tilde{P}$. Applying Girsanov’s Theorem for semimartingales ([JS87], Chapter III, Theorem 3.24), it is possible to obtain an explicit expression of $\left( \tilde{A}, \tilde{C}, \tilde{\nu} \right)$ starting from a “good” version $(A, C, \nu)$ of the characteristic of $X$ with respect to $P$.

Let $P$ be the predictable $\sigma$-algebra.

**Proposition A-1.1** The characteristics of $X$ under a probability measure $\tilde{P}$, equivalent to $P$, is given by the triple $\left( \tilde{A}, \tilde{C}, \tilde{\nu} \right)$:

$$
\tilde{A} = \begin{pmatrix}
\int_{[0,t] \times \mathbb{R}} \pi \, d\tilde{\nu} \\
\int_{[0,t] \times \mathbb{R}} r \, d\tilde{\nu} \\
\int_0^t [k^{sh} (b(R_s) - R^{sh}_s) + \sigma^2 (|R_s - R^{sh}_s|) R^{sh}_s \gamma_s] \, ds
\end{pmatrix} = \begin{pmatrix}
\tilde{A}^\pi \\
\tilde{A}^R \\
\tilde{A}^{sh}
\end{pmatrix},
$$

(83)

and

$$
\tilde{C} = C,
$$

(84)

and

$$
\tilde{\nu}(dt, d\pi, dr, dz) := \left\{ Y(t, (\pi \ 0 \ 0)^t) \left[ N(d\pi, a_{t-}) \times \delta_0(dr) \times \delta_0(dz) \right] \mathbb{1}_T(t) + \right.
\left. Y(t, (0 \ r \ 0)^t) \left[ \delta_0(d\pi) \times \sum_{l=-m, m \neq 0} \mathbb{P}(\delta_{-l}(dr) \lambda^P(\Pi_t, R_t)) + \delta_l(dr) \lambda^{U}(\Pi_t, R_t)) \times \delta_0(dz) \right] \mathbb{1}_{[0,T]}(t) \right\} dD_t,
$$

(85)

where $Y : \Omega \times \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^+$ is $P \times \mathcal{B}(\mathbb{R}^3)$-measurable, $\gamma$ is a predictable process, and $Y$ and $\gamma$ satisfy

$$
\int_{[0,t] \times \mathbb{R}^3} |x(Y(\omega, t, x) - 1)| \, d\nu < \infty,
$$

(86)

$$
\int_0^t \sigma^2 (|R_s - R^{sh}_s|) R^{sh}_s |\gamma_s| ds < \infty,
$$

(87)

$$
\int_0^t \sigma^2 (|R_s - R^{sh}_s|) R^{sh}_s \gamma_s^2 ds < \infty,
$$

(88)

for all $t \in \mathbb{R}^+$ and $\tilde{P}$-a.s.

**Proof.** The assertion follows from Theorem 3.24 of Jacod-Shiryaev [JS87]. □

Among the infinitely many measures $\tilde{P}$ equivalent to $P$, we choose the measures under which the dynamics of $X$ has the same structure under $P$ with different parameters. Recall that $\mathcal{F} = \{ t_i \}_{i=0, \ldots, M}$. 

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Proposition A-1.2 Let $\widetilde{P}$ be a probability measure equivalent to $P$ under which the characteristics of $X$ are given by (83)-(84) and (85), with parameters $\tilde{\alpha}, \tilde{\beta}, \tilde{\eta}$ ($l = 1, \ldots, m$), $\tilde{c}^U, \tilde{c}^D, \tilde{k}^{sh}, \tilde{b}_0, \tilde{b}_1$. Then the function $Y$ and the process $\gamma$ of Proposition A-1.1 are given by

\[
Y(t, (\pi 0 0)^t) = \exp \left\{ \frac{(a_1-\bar{a}_1)(a_2-\bar{a}_2)-2\pi}{2a_2} \right\}, \quad t \in \mathcal{T}
\]

\[
Y(t, (0 r 0)^t) = \sum_{l=-m, l \neq 0}^{m} \frac{\tilde{p}_l(\delta_l(\eta))\gamma^3(\Pi_l, R_t) + \delta_l(\eta)\lambda^l(\Pi_l, R_t)}{\gamma^{l+1}(\Pi_l, R_t)} + \frac{\tilde{c}^l}{2} \int_t^T \phi \left( \frac{X_s^r}{X_s^l} \right) ds, \quad t \in \mathbb{R}_+ - \mathcal{T}, \quad (89)
\]

\[
\gamma_t = \frac{\tilde{k}^{sh}(\tilde{b}(R_t) - R_t^{sh}) - \tilde{k}^{sh}(\tilde{b}(R_t) - R_t^{sh})}{\tilde{c}^{sh}(\tilde{b}(R_t) - R_t^{sh})}, \quad t \in [0, T].
\]

**Proof.** The assertion follows by direct computation from Proposition A-1.1. □

**Proof of Theorem 4.1.** Setting $\widetilde{R}_t = \int_0^t R_s^{sh} ds$ and applying the generalized Itô's formula to (19), one gets, for $t_i \leq t < t_{i+1}$,

\[
P_t = P_{t_i} + \int_{t_i}^{t} \exp \left\{ \widetilde{R}_s \right\} M_s d\widetilde{R}_s + \int_{t_i}^{t} \exp \left\{ \widetilde{R}_s \right\} dM_s = P_{t_i} + \int_{t_i}^{t} P_t R_s^{sh} ds + \int_{t_i}^{t} \exp \left\{ \widetilde{R}_s \right\} dM_s. \quad (90)
\]

On the other hand, applying the generalized Itô's formula to (20), we have, for $t_i \leq t < t_{i+1}$, $i = 0, \ldots, M - 1$,

\[
P_t = P_{t_i} + \int_{t_i}^{t} \partial_s \varphi(s, X_s) ds + \int_{t_i}^{t} \nabla \varphi(s, X_s^-) dX_s + \sum_{t_i < s \leq t} [\varphi(s, X_s) - \varphi(s, X_s^-) - \nabla \varphi(s, X_s^-) \Delta X_s] + \frac{1}{2} \int_{t_i}^{t} \text{tr} \left( H \varphi(s, X_s) d[X^c, X^c]_s \right), \quad (91)
\]

and, at $t_i$,

\[
P_{t_{i+1}} - P_{t_{i+1}} = \varphi(t_{i+1}, X_{t_{i+1}}) - \varphi(t_{i+1}, X_{t_{i+1}}), \quad i = 0, \ldots, M - 1,
\]

\[
P_T = \Phi^M(T, X_T). \quad (92)
\]

Consider (91): using the canonical decomposition of each component of $X$ with respect to $\widetilde{P}$, we have

\[
P_t = P_{t_i} + \int_{t_i}^{t} \partial_s \varphi(s, X_s) ds + \int_{t_i}^{t} \left( \frac{\partial \varphi}{\partial X_s}(s, X_s^-) d\widetilde{A}_1 + \frac{\partial \varphi}{\partial X_s}(s, X_s^-) d\widetilde{A}_2 + \frac{\partial \varphi}{\partial X_s}(s, X_s^-) d\widetilde{A}_3 + \sum_{t_i < s \leq t} [\varphi(s, X_s) - \varphi(s, X_s^-)] - \sum_{t_i < s \leq t} \left( \frac{\partial \varphi}{\partial X_s}(s, X_s^-) \Delta X_1 + \frac{\partial \varphi}{\partial X_s}(s, X_s^-) \Delta X_2 + \frac{\partial \varphi}{\partial X_s}(s, X_s^-) \Delta X_3 + \frac{1}{2} \int_{t_i}^{s} \text{tr} \left( H \varphi(s, X_s) d[X^c, X^c]_s \right) + d\tilde{M}_t, \quad (93)
\]

where $\tilde{M}_t$ is the martingale part of $X$. Note that $\Delta X_{3s} = 0$ because $X_3 = R^{sh}$ is a continuous process and that the terms in $d\tilde{A}_1$ are equal to the terms in $\Delta X_1$, for $i = 1, 2.$
Moreover we have \( d[X^c, X^c]_t = \sigma^2(|R_s - R_s^{sh}|)R_s^{sh} \) and replacing \( \tilde{A}_{3s} = \tilde{A}_{s}^{sh} \) with its explicit expression given by (83), one gets

\[
P_t = P_{t_i} + \int_{t_i}^t \partial_s \varphi(s, X_s) ds + \int_{t_i}^t \int_{t_{i+1}}^{t_{i+1}} \partial_t \varphi(s, X_{s-}) k^{sh} (b(R_s) - R_s^{sh}) ds + \int_{t_i}^t \int_{t_{i+1}}^{t_{i+1}} \partial_t \varphi(s, X_{s-}) \lambda_D (\Pi_{s-}, R_{s-}) ds + \frac{1}{2} \int_t^{t_i} \sigma^2(|R_s - R_s^{sh}|) R_s^{sh} \varphi(s, X_s) ds + \hat{d} \hat{M}_t.\]

Now, \( \hat{P} \text{-a.s.} \) (90) and (94) must be equal, hence, disregarding both martingale parts, for \( t \in [t_i, t_{i+1}) \) we have

\[
\varphi(t, \Pi_t, R_t^{sh}) = \frac{\partial \varphi(t, \Pi_t, R_t^{sh})}{\partial t} + k^{sh} (b(R_t) - R_t^{sh}) \frac{\partial \varphi(t, \Pi_t, R_t^{sh})}{\partial z} \bigg|_{\Pi_t = R_t, z = R_t^{sh}} + \frac{1}{2} \sigma^2 (|R_t - R_t^{sh}|) R_t^{sh} \frac{\partial^2 \varphi(t, \Pi_t, R_t^{sh})}{\partial z^2} \bigg|_{\Pi_t = R_t, z = R_t^{sh}} + \sum_{l=-m}^{m} \left( \varphi(t, \Pi_t, R_t^{sh} + y_l) - \varphi(t, \Pi_t, R_t^{sh}) \right) \lambda_U (\Pi_t, R_t) + \sum_{l=-m}^{m} \left( \varphi(t, \Pi_t, R_t^{sh} - y_l) - \varphi(t, \Pi_t, R_t^{sh}) \right) \lambda_U (\Pi_t, R_t).\]

At \( t_{i+1} \), taking into account that, \( \hat{P} \text{-a.s.} \) \( \Pi_{t_{i+1}} = \Pi_{t_i}, R_{t_{i+1}} = R_{t_i}^{sh} + 1 \) and \( \hat{P} \text{-a.s.} \) \( \Pi_{t_{i+1}} = \Pi_{t_i}, R_{t_{i+1}} = R_{t_i}^{sh} + 1 \), one gets

\[
\varphi(t_{i+1}, \Pi_{t_i}, R_{t_i}^{sh}, R_{t_{i+1}}^{sh}) = \frac{1}{\sqrt{2\pi v}} \int_{\mathbb{R}} \varphi(t_{i+1}, a_{t_{i+1}} + u, R_{t_{i+1}}^{sh}) \exp \left(-\frac{u^2}{2v^2}\right) du, \tag{96}\]

where \( a_{t_{i+1}} = (\alpha - k\Pi - 1)\Pi_t + k\Pi \pi^* + \beta \left( R_{t_{i+1}} - R_{t_{i+1}}^{sh} \right) \). Therefore, on each interval \([t_i, t_{i+1})\), the function \( \varphi \) is given by the function \( \varphi^t \) that solves the terminal value problem (22)-(23)-(24).

\[\square\]

**Lemma A-1.6** The function \( \varphi \) defined in (32) is Hölder continuous in \( t \), with exponent \( \frac{1}{2} \), uniformly for \((r, z) \in [0, \overline{r}] \times [l, L] \).

**Proof.** \( \varphi \) can be written as

\[
\varphi(t, r, z) = E \left[ \Phi \left( R_{t_{i+1}}^{sh, r, z} \right) e^{-f_{t_{i+1}}^{(t_{i+1})} R_{s}^{sh, r, z} ds} \right],
\]

where \((R_{t_i}^{r, z}, R_{t_i}^{sh, r, z})\) is the solution of

\[
\begin{align*}
R_{t_i}^{r, z} &= r + \int_0^t J \left( \pi, R_{s}^{r, z}, \zeta_{N_{s} - 1}, \lambda_D \right) dN_s, \\
R_{t_i}^{sh, r, z} &= z + \int_0^t k^{sh} \left( b(R_{s}^{sh, r, z}) - R_{s}^{sh, r, z} \right) ds + \int_0^t \sigma \left( |R_{s}^{sh, r, z} - R_{s}^{sh, r, z}| \right) \sqrt{R_{s}^{sh, r, z}} dW_s.
\end{align*}
\]
Then, for $s \leq t$,
\[
|\varphi(t_1 - t, r, z) - \varphi(t_1 - s, r, z)| \leq c \mathbb{E} \left[ \int_s^t R_{u_s}^{sh, r, z} \, du + |R^{r, z}_t - R^{r, z}_s| + |R^{sh, r, z}_t - R^{sh, r, z}_s| \right],
\]
for some $c > 0$.

Therefore
\[
|\psi(t_1 - t, r, z) - \psi(t_1 - s, r, z)| \leq c \left\{ \mathbb{E} \left[ \int_s^t R_{u_t}^{sh, r, z} \, du + \int_s^t \left| J(t, R_{u_t}^{r, z}), \zeta_{N_{u_t}} \right| \, dN_u + \int_s^t k^{sh} \left( b(R_{u_t}^{r, z}) - R_{u_t}^{sh, r, z} \right) \, du \right] + \mathbb{E} \left[ \left( |R^{r, z}_t - R^{sh, r, z}_u| \sqrt{R^{sh, r, z}_u} \, dW_u \right)^2 \right]^{1/2} \right\} \leq c \left\{ \int_s^t \sum_{i=-m, m \neq 0} y_{d_i} \delta + k^{sh} (b(\varphi)) + (1 + k^{sh})^2 \mathbb{E} \left[ R_{u_t}^{sh, r, z} \right] \, du + \int_s^t \sigma^2_1 (1 + \varphi) \mathbb{E} \left[ R_{u_t}^{sh, r, z} \right] + \sigma^2_1 \mathbb{E} \left[ \left( R_{u_t}^{sh, r, z} \right)^2 \right] \right\}^{1/2},
\]
and the assertion follows from the fact that
\[
\sup_{[0, \varphi]} \sup_{0 \leq t \leq t_1} \mathbb{E} \left[ \left( R_{u_t}^{sh, r, z} \right)^2 \right] < \infty.
\]

**Lemma A-1.7** For fixed $r \in [0, \varphi]$, the initial-boundary value problem
\[
\begin{aligned}
\frac{\partial u}{\partial t}(t, z) &= A^r u(t, z) = g(t, z), \quad t \in (0, t_1], \quad z \in (l, L), \\
u(0, z) &= h(0, z), \quad z \in (l, L), \\
u(t, l) &= h(t, l), \quad t \in (0, t_1], \\
u(t, L) &= h(t, L), \quad t \in (0, t_1],
\end{aligned}
\]
where
\[
A^r f(z) = k^{sh} (b(r) - z) f'(z) + \frac{1}{2} \sigma(|r - z|) z f''(z),
\]
has one and only one solution in $C^{1,2}((0, t_1] \times (l, L)) \cap C([0, t_1] \times [l, L])$, for any $g \in C((0, t_1] \times [l, L])$ Hölder continuous of order $p$ in $t$, uniformly with respect to $z$, and any $h \in C((0, t_1] \times [l, L])$.
Proof. Since \( r \) is fixed, we will omit the dependence upon \( r \). By Theorem 9, Chapter 3 of Friedman [FRI64], it is enough to consider the case \( h = 0 \). For smooth \( g \), the solution is given by

\[
  u(t, z) = \int_0^t ds \int_{(l,L)} dy G(t, z, s, y)g(s, y),
\]

where \( G \) is the Green function (see Garroni and Menaldi [GM92]).

Now let \( g \) be as in the assertion and let \( u \) be still defined by (101). Consider a sequence \( \{g^n\} \) of smooth functions converging to \( g \) uniformly over \([0,t_1] \times [l,L]\) and such that

\[
  \sup_n \sup_{x \in [l,L]} |g^n(t, z) - g^n(s, z)| \leq H|t - s|^p
\]

for some \( H > 0 \). Let \( u^n \) denote the corresponding solutions of (100) (with \( h = 0 \)). Then, by exploiting the estimates of Garroni and Menaldi [GM92] on \( G \), one can see that

- \( u^n(t, z) \rightarrow u(t, z) \),
- \( \frac{\partial u^n}{\partial t}(t, z) \rightarrow \int_0^t ds \int_{(l,L)} dy \frac{\partial G}{\partial t}(t, z, s, y) [g(s, y) - g(t, y)] + \int_{(l,L)} dy G(t, z, 0, y)g(t, y) \),
- \( \frac{\partial u^n}{\partial z}(t, z) \rightarrow \int_0^t ds \int_{(l,L)} dy \frac{\partial G}{\partial z}(t, z, s, y)g(s, y) \),

as \( n \rightarrow \infty \), uniformly over compact subsets of \((0,t_1] \times (l,L)\), which yields the assertion. \( \square \)
References


