Complete duality for quasiconvex dynamic risk measures

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Quasiconvex analysis has important applications in several optimization problems in science, economics and finance. Recently, a Decision Theory complete duality involving quasiconvex real valued functions has been proposed by [1]. Important inputs to finance has been given by Maccheroni et al. [2] and Kupper et al. [3] in the theory of Risk Measures. During financial crisis a lack of liquidity in the market may cause many troubles in covering the losses and consequently cash additivity must be dropped. Giving up the cash additivity of risk measures, convexity and quasiconvexity are not anymore equivalent: the authors in [2] argue that from a theoretical point of view quasiconvexity gives a better explanation and description of the diversification principle.

A function $f : L \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ defined on a vector space L is quasiconvex if for all $c \in \mathbb{R}$ the lower level sets $\{X \in L \mid f(X) \leq c\}$ are convex. In a general setting, the dual representation of such functions was shown by Penot and Volle.

To the best of our knowledge, a *conditional* version of this representation was lacking in the literature. When $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space, many problems having dynamic features leads to the analysis of maps $\pi : L_t \to L_s$ between the sets of random variables $L_t \subseteq L^0(\Omega, \mathcal{F}_t, \mathbb{P})$ and $L_s \subseteq \bar{L}^0(\Omega, \mathcal{F}_s, \mathbb{P})$, $0 \leq s < t$.

In this talk we consider quasiconvex maps of this form and analyze their dual representation, comparing two different possible approaches. The aim is to present a conditional version of the dual representation, when L_t is a **topological vector space** or a **topological L^0 module**. The weak assumptions on L_t allow us to consider maps π defined on the typical spaces used in the literature in this framework:

- L_t is a topological vector space, such as $L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$, $L^p(\Omega, \mathcal{F}_t, \mathbb{P})$, the Orlicz spaces $L^\Psi(\Omega, \mathcal{F}_t, \mathbb{P})$, $M^\Psi(\Omega, \mathcal{F}_t, \mathbb{P})$.
- L_t is a topological L^0-module, such as $L^p_{\mathcal{G}}(\mathcal{F})$.

These two algebraic structures are deeply different and the study of linear functionals and dual spaces has little in common. For this reason we need *ad hoc* arguments for each case in order to reach a robust representation of the form

$$\pi(X) = ess \sup_{Q \in \mathcal{P}_t} R(E_Q[X|\mathcal{F}_s], Q), \tag{1}$$

where

$$R(Y, Q) := ess \inf_{\xi \in L_t} \{\pi(\xi) \mid E_Q[\xi|\mathcal{F}_s] \geq_Q Y\}, \ Y \in L_s,$$

and $\mathcal{P}_t$ is an opportune set of probabilities measures.

In the particular case of quasiconvex conditional maps defined on L^0 modules of the L^p type we provide a complete duality matching the results holding true for the static case as shown by Cerreia Vioglio et al. in [2].

References

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