Convex Rearrangements

For a process $X(t),\ t \geq 0$ with stationary increments, observed at times $0, 1/N, \ldots, (N-1)/N$ define increments

$$Y_{k,N} = X((k+1)/N) - X(k/N),\ k = 0, 1, \ldots, N-1. \tag{1}$$

Let, further $Y_{0:N} \leq Y_{1:N} \leq \cdots \leq Y_{(N-1):N}$ denote the order statistic of Y's from (1).

A *convex rearrangement* of $X(t)$ is defined as

$$\mathcal{V}X_N(t) = X(0) + \sum_{i=0}^{[Nt]-1} Y_{i:N} + (Nt - \lfloor Nt \rfloor) Y_{\lfloor Nt \rfloor : N},$$

The sequence $Y_{k,N},\ k = 0, 1, \ldots, N-1$ is stationary with variance $\sigma_N^2 = \mathbb{E}Y_{1,N}^2$ and covariance $\gamma_N(h) = \mathbb{E}Y_{k,N}\, Y_{k+h,N}$.

Davydov (1998), Davydov and Thilly (2000), and Phillippe and Thilly (2002) investigated the asymptotic behavior of normalized $\mathcal{V}X_N(t)$ for broad classes of Gaussian and α-stable processes.

Theorem 0.1 *(*DAVYDOV AND THILLY, 2000*)*

Let for each $h \in \{1, 2 \ldots, N-1\}$

$$[\text{CONDITION } \mathcal{A}] \quad \left| \frac{\gamma_N(h)}{\sigma_N^2} \right| \leq \min\{1, K \cdot (\ln h)^{-1-\tau}\},$$

for some fixed positive constants K *and* τ. *Denote by*

$$L(t) = \int_0^t \Phi^{-1}(s) ds = -\frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left(\Phi^{-1}(t) \right)^2 \right\},$$

where Φ *is the c.d.f. of a standard normal distribution. Then, when* $N \to \infty$,

$$\frac{\mathcal{V}X_N(t)}{b_N} \to L(t), a.s.$$

where $b_N = N\sigma_N$.

Example 0.1 Let $B_H(t)$ be the fractional Brownian motion with the exponent H and $\mathcal{V}B_{H,N}(t)$ it convex rearrangement. Since $\sigma_N^2 = \mathbb{E}Y(1,N)^2 = \mathbb{E}B_H(1/N)^2 = C \cdot (1/N)^{2H}$. Then, the condition $\mathcal{A}$ in Theorem 0.1 is satisfied and when $N \to \infty$ almost surely on [0,1],

$$\frac{1}{N^{1-H}\sqrt{C}}\mathcal{V}B_{H,N}(t) \to L(t). \tag{2}$$

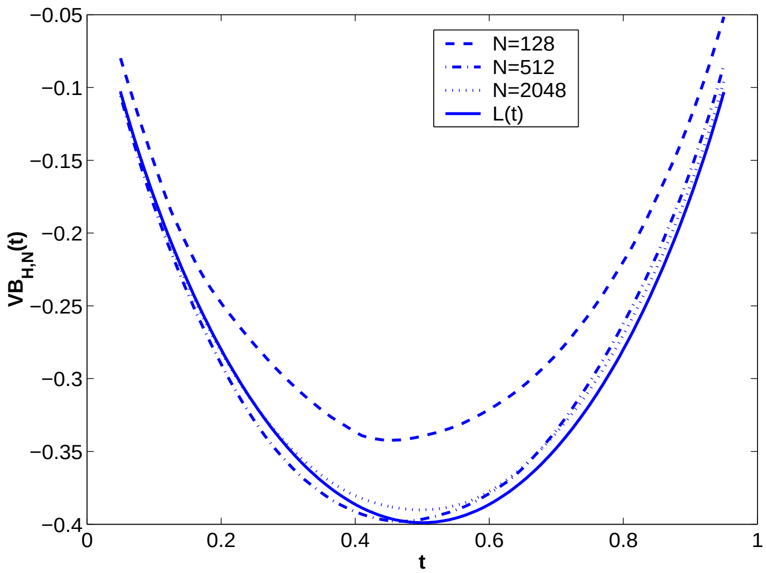

Figure 1: Apprpximation of $\int_0^t \Phi^{-1}(s)ds$ (solid line) by normalized $\mathcal{V}B_{H,N}$ for $H = 0.4$ and $N = 128$ (–), 512 (-.), and 2048. (..)

By solving (2) with respect to H, on can propose a family of consistent estimators of H, indexed by a continuous parameter $t_0 \in$

$(0,1)$. That opens an interesting question of design which t_0 to select.

Let the self-similar process $X(t),\ t \geq 0$ satisfies the hypothesis $\mathcal{A}$ in Theorem 0.1. Let $t_0 \in (0,1)$ and let

$$\hat{E}(N, t_0) = 1 - \frac{\log |\mathcal{V} X_N(t_0)/\sqrt{C} L(t_0)|}{\log n}.$$

Then $\hat{E}(N, t_0) \longrightarrow H,\ a.s.$, when $N \to \infty$.

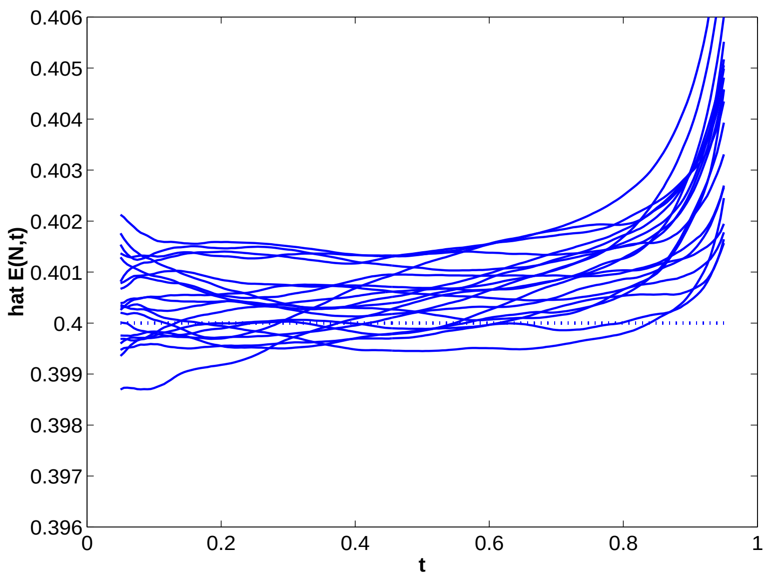

Figure 2: Estimators $\hat{E}(N,t)$ for $H = 0.4$, and $N = 2096$

Let $X(t),\ t \geq 0$ be a selfsimilar process satisfying $\mathcal{A}$ and let $\boldsymbol{g}$ be the filter. If the convex rearrangements are applied on the $\boldsymbol{g}$-filtered process $Y\left(\frac{k}{N}\right) = \sum_{n=0}^{\ell} g_n X\left(\frac{k-n}{N}\right),\ k = 0, 1, \ldots, N-1$ the resulting polygonal process $Y_N(t) = Y\left(\frac{k}{N}\right) + (Nt-k)\left[Y\left(\frac{k-1}{N}\right) - Y\left(\frac{k}{N}\right)\right]$, satisfies the condition $\mathcal{A}$ as well, and

$$\frac{1}{b_N} \mathcal{V} Y_N(t) \longrightarrow L(t), a.s., \text{ when } N \to \infty, \qquad (3)$$

where $b_N = N\sigma_N(\boldsymbol{g})$. The filter $\boldsymbol{g}$ of length ℓ has to be selected in such a way so that the variance $\sigma_N(\boldsymbol{g}) = \mathbb{E}\left|Y(\frac{k-1}{N}) - Y(\frac{k}{N})\right|^2$ is not 0. Sufficient condition is that $f(s) = \sum_{i,j=0}^{\ell} g_i g_j |i-j|^s$ does not vanish for $s \in (0,2)$ for the selected filter $\boldsymbol{g}$.

The double filtering of $X(t)$ in (3) can be replaced by a single filter $\boldsymbol{g}$ with properties $\sum_n n^r g_n = 0,\ 0 \le r \le p-1$ and $\sum_n n^p g_n \neq 0$. Indeed, only the existence of first vanishing moment allows the decomposition of $\boldsymbol{g}$ to convolution of $\{1,-1\}$ and a finite filter $\boldsymbol{u}$ of length $\ell - 1$.

Thus denote ***g**-convex rearrangement* of $X(t)$ as

$$\mathcal{V}X_{\boldsymbol{g},N}(t) = X(0) + \sum_{i=0}^{[Nt]-1} Y_{\boldsymbol{g},(i:N)} + (Nt - \lfloor Nt \rfloor) Y_{\boldsymbol{g},(\lfloor Nt \rfloor:N)},$$

where $\{Y_{\boldsymbol{g},(0:N)}, \ldots, Y_{\boldsymbol{g},(N-1:N)}\}$ is order statistics for the sequence $Y_{\boldsymbol{g}}\left(\frac{k}{N}\right) = \sum_{n=0}^{\ell} g_n X\left(\frac{k-n}{N}\right)$, for $k = 0, 1, \ldots, N-1$.

Let $\boldsymbol{g}^d$ be dilation of filter $\boldsymbol{g}$ obtained by inserting $d-1$ zeros between non-zero filter taps. For example, for $\boldsymbol{g} = \{1\ \ -2\ \ 1\}$ the 3-dilated filter is $\boldsymbol{g} = \{1\ \ 0\ \ 0\ \ -2\ \ 0\ \ 0\ \ 1\}$.

Theorem 0.2

Let

$$\mathbb{D}(N, d_1, d_2, t_0) = \frac{\mathcal{V}X_{\boldsymbol{g}^{d_2},N}(t_0)}{\mathcal{V}X_{\boldsymbol{g}^{d_1},N}(t_0)}. \tag{4}$$

Then for any $t_0 \in [0,1]$, *and integers* d_1, *and* d_2,

$$\frac{\log |\mathbb{D}(N, d_1, d_2, t_0)|}{\log(d_2/d_1)} \longrightarrow H,\ a.s.$$

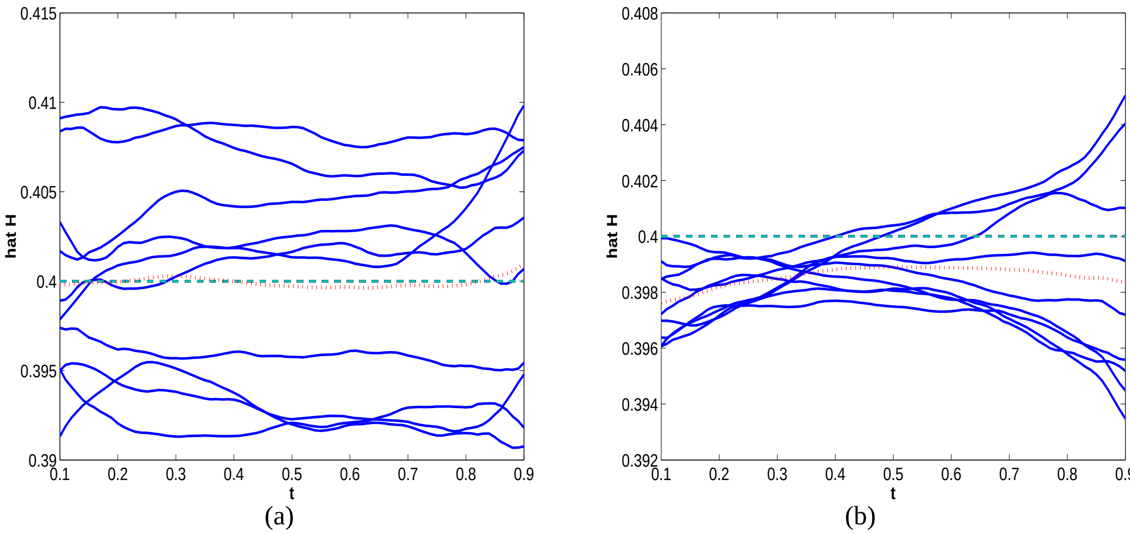

Figure 3: Estimation of H by (4). See the figconcre3.m

APPENDIX: MATLAB PROGRAMS

```
function  filtd = dilate_filter(filt,k)
%-------------------------------------
%  Dilate Filter by k zeros between the original
% taps, k integer > 0.
%-------------------------------------
newlength = (k+1)*length(filt)-k; filtd = zeros(1,newlength);
filtd(1:(k+1):newlength)=filt;
%--------------------------------------
% Brani 10/15/02
```

```
function v = conre(t, Proc, fi)
%----------------------------
% Convex Rearrangement
% Proc can be, for example MakeFBMNew(1024, 1/3, 1);
n = length(Proc);
difs =filter(fi,1,Proc);
difs = difs(length(fi):n);
nn = length(difs);
diffs(1:nn)=sort(difs);
upper = floor(nn.*t); v=[];
for i = 1:length(t)
 v = [v sum( diffs(1:(upper(i)-1)))+ ...
   ((nn.*t(i) - upper(i)).*diffs(floor(nn.*t(i))))];
end
```

```
 %---------------------------------------------
 close all
 clear all
 family = 'Symmlet'; par = 4;
 k1 = 2; k2 = 3; H= 0.4; seed = 1;
 n = 128 * 1024;
 b = (0.1:0.01:0.9);
 baseline = [];
 for seed = 1:5
 %---------------------------------------------
 a = MakeFBMNew(n, H, seed);
 filt = MirrorFilt(MakeONFilter(family, par));
 filtd1 = dilate_filter(filt,k1);
 filtd2 = dilate_filter(filt,k2);
 y1 = conre(b,a, filtd1);
 y2 = conre(b,a, filtd2);
 z = y2./y1;
 zz = log2( abs(z) )./log2((k2+1)/(k1+1));
 plot(zz)
   hold on
 plot(H*ones(length(zz)))
   hold on
 baseline = [baseline; zz];
 end
plot(mean(baseline),'r-')
```

REFERENCES

Davydov Yu., Thilly E., Convex rearrangements of Gaussian processes. Theory of Probability and its Applications.

Philippe, A. and Thilly, E., Identification of locally self-similar Gaussian process by using convex rearrangements, Methodology and Computing in Applied Probability Vol 4, N2, p 195-207, (2002)