CNR-IMATI Milano

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WAVELETS AND SELF-SIMILARITY: THEORY AND APPLICATIONS

Lecture 7: Multifractals

P L A N

1. **Multifractal Spectra**

2. **Bin's Three Point Summary of Multifractal Spectra**

Wavelet-based Multifractal Spectrum

- The wavelet-based calculation of a multifractal spectrum depends on the concepts of *partition function* and *Legendre transform.*
- The partition function $T(q)$ is defined using wavelets as

$$T(q) = \lim_{j \to -\infty} \log_2 E|d_{j,k}|^q,$$

where $d_{j,k}$ is the L_1-normalized wavelet coefficient at level j and position index k, and q is the moment-order.

- Normalization: $\psi_{j,k}(x) = 2^{-j}\psi(2^{-j}x - k)$.
- Parameter q is real and can be positive or negative. However, the interpretation of negative moments is still not clear (or physical).
- Even though partition function is informative, the singularity measure is not explicit. It has been proposed by Arneodo and his team in the early 1990's that the local singularity strength can be measured in terms wavelet coefficients as:

$$\alpha(t) = \lim_{k2^j \to t} \frac{1}{j} \log_2 |d_{j,k}|$$

■ It has been shown (Jaffard, 1995) that the wavelet coefficients preserve the scaling behavior (global + local) of the process conditionally the wavelet is more regular than the process itself.

■ $\alpha(t)$ measures the "intensity of oscillations" of the process at time t. Small values of $\alpha(t)$ reflect more irregular behavior at time t.

■ Any process path has a collection of local singularity strength measures and their distribution $f(\alpha)$ forms the multifractal spectrum. A direct way to obtain this spectrum is to use the counting technique,

$$f(\alpha) = \lim_{\epsilon \to 0} \#\{\alpha(t) : \alpha - \epsilon < \alpha(t) < \alpha + \epsilon, -\infty < t < \infty\}.$$

■ This method is not practicable due to the difficulty of approximating the limit. A useful tool to improve the estimation efficiency is the Legendre transform. The Legendre transform of the partition function is

$$f_L(\alpha) = \inf_q \{q\alpha - T(q)\}.$$

It can be shown that $f_L(\alpha)$ converges to the true multifractal spectrum using the theory of large deviations.

mfspectra

```
function [f,alfa]=mfs(data,q,j1,j2,L,wavelet,par)
%function [f,alfa]=mfs(data,q,j1,j2,L,wavelet,par)
%
%   Input:data---the input signal vector;
%         q------the range of moment orders (usually -1...6, or similar)
%         j1,j2-----the range of interest scale(min and max)
%         L,wavelet,par----parameters for wavlet decomposition
%   Output: f----multifractal spectrum f(alpha)
%   For example: [a,b]=mfs(m,-1:0.2:6,3,10);
%   where m is the fractal signal, such as fbm(1/3) with 2^16 length.
%   q=[-1,6] with equal space 0.2. [j1, j2]=[3,10]
%   reference:P. Goncalves, R. H. Riedi and R. G. Baraniuk
%            Simple Statistical Analysis of Wavelet-based
%            Multifractal Spectrum Estimation,
%            Proceedings of the 32nd Conference on ‘Signals,
%            Systems and Computers’,   Asilomar, Nov 1998
```

```
if nargin == 4, wavelet='Daubechies'; par=20; L=2;end;  %defaults
nn=length(data);
lnn= log2(nn);
% need active path to WaveLab801 at this point!!!
wf = MakeONFilter(wavelet, par);
wddata = FWT_PO(data, L, wf);
for i =  L:(lnn-1)
    j=lnn-i;
    help = 2^(-j/2)*wddata(dyad(i)); %L1 normalization
    for k=1:length(q)
        s(j,k)=mean(abs(help).^q(k)); %
    end;
end;
t=[];
for k=1:length(q)
a=polyfit(j1:j2,log2(s(j1:j2,k))',1); %regression
t=[t,a(1)];
end;
alfa=diff(t)./diff(q); % numerical derivative
qq=q(1:length(q)-1);
f=qq.*alfa-t(1:length(q)-1);
```

■ From the practical point of view, we still require a robust estimator of the partition function.

$$E|d_{j,k}|^q \sim 2^{jT(q)}, \quad \text{as} \quad j \to -\infty$$

■ It has been shown that the qth moment of the wavelet coefficients of the power law process (Arneodo, 1998) satisfies the following equation:

$$E|d_{j,k}|^q = C_q 2^{jqH}$$

where H is the self-similarity exponent and C_q is a constant depending only on q.

■ It is a standard practice to use linear regression to identify the self-similarity exponent H since the values $E|d_{j,k}|^q$ could be easily obtained by moment-matching method thereby facilitating the estimation of $T(q)$.

$$\log_2 \widehat{S_j(q)} \sim jT(q) + \varepsilon_j,$$

where $\widehat{S_j(q)} = \frac{1}{N2^{-j}} \sum_{k=1}^{N2^{-j}} |d_{j,k}|^q$ is the empirical q^{th} moment of the wavelet coefficients (N is the length of the entire time series) and the error term ε_j is introduced from the moment matching method when replacing the true moments with the empirical ones. Simple ordinary least square (OLS) is the most convenient choice of estimating the partition function.

■ The bias can be very large in some extreme cases since the variance of the empirical q^{th} moments is not constant with respect to scale j. The variance of $\log_2 \widehat{S_j(q)}$ is

$$Var(\log_2 \widehat{S_j(q)}) = \frac{A(q)}{N2^{-j}} + \frac{B(q)}{N^2 4^{-j}} + \cdots$$

where $A(q)$ and $B(q)$ are constants depending only on the underlying distribution function of the finest wavelet coefficient.

■ The regression problem is a heteroskedastic problem in which the variances of the error terms are not constant across the scales. Even though the OLS solution of $T(q)$ is still unbiased and consistent asymptotically, it is no longer efficient due to the heteroskedasticity.

■ Weighted least squares (WLS) is used to obtain efficient unbiased estimates. WLS estimator downweights the squared residuals for scales with large variances, in proportion to those variances. If one finds $w_j = Var^{-1}(\log_2 \widehat{S_j(q)})$, a WLS estimator of $T(q)$ is given by

$$\widehat{T(q)} = \frac{\sum_{j=1}^J w_j \sum_{j=1}^J j w_j \log_2 \widehat{S_j(q)} - \sum_{j=1}^J j w_j \sum_{j=1}^J w_j \log_2 \widehat{S_j(q)}}{\sum_{j=1}^J w_j \sum_{j=1}^J j^2 w_j - (\sum_{j=1}^J j w_j)^2}$$

■ In practice, the exact analytical formula $Var(\log_2 \widehat{S_j(q)})$ is too complicated to be used directly. However, if the N is reasonably large it is natural to use the approximate weights $p_j = N2^{-j}$.

■ This WLS estimator results in a variance given by,

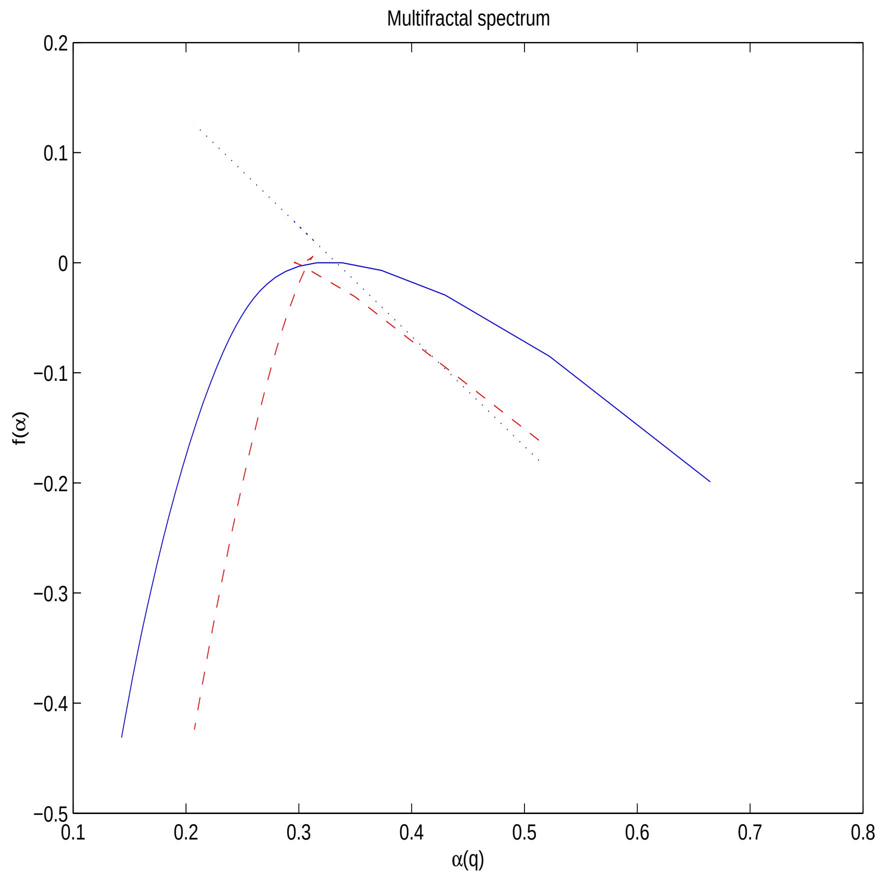
$$Var(\widehat{T(q)}) = \frac{A(q)C(J)}{N} + \frac{B(q)D(J)}{N^2}$$

where the constants $C(J)$ and $D(J)$ can be evaluated from the formula provided in [**?**].

■ Once the $T(q)$ is estimated, the next step is to perform the Legendre transform. Since $\frac{\partial(\alpha q - T(q))}{\partial q} = \alpha - T'(q)$ and $T''(q) < 0$ (Goncalves, 1998), the maximum value of $\alpha q - T(q)$ is achieved at $q = T'^{(-1)}(\alpha)$. So, performing the Legendre transform is divided into two steps: First, the numerical derivative of $T(q)$ is obtained using the finite difference; then, the value of Legendre spectrum at $\alpha = \widehat{T'(q)}$ is evaluated. We point out that the Legendre transform is not able to estimate the multifractal spectrum value at arbitrary singularity strength α. The set of the multifractal spectrum values is determined by a pre-specified vector of q values. The more q values used, the finer the multifractal spectrum will appear, i.e., the resolution of the spectrum is determined by the "(order) sampling frequency" of the moments.

Geometric Attributes of the Multifractal Spectrum

- Theoretically, the multifractal spectrum of a fractional Brownian Motion or fBm process (representative of mono-fractals) consists of three geometric parts: the vertical line, the maximum point and the right slope.
- The maximum point corresponds to the Hurst exponent and the vertical line and the right slope are thought to be inherent features, which distinguish fBm from a multifractal process. However, it is impossible spectrum in practice. Even for a well simulated fBm, due to finite sample size and estimation error (the partition function estimation and derivative approximation are responsible for most of the errors), its spectrum may deviate from the theoretical form
- Comparing with the turbulence measurement, the fBm is much closer to the vertical line and this closeness may be quantified by the left slope of the spectra. Another important difference between these two spectra is the width spread of the spectra. It is obvious that the width spread of the fBm is much smaller than that of the turbulence measurement indicating lack of richness in singularity indices for the fBm.



The spectrum can be approximately summarized by the "left" slope, the maximum point, and the width spread.

- The left and right slopes can be obtained using linear regression.
- **Definition:** Suppose that α_1 and α_2 are two roots that satisfy the equation

$f(\alpha) + d = 0$ and $\alpha_1 < \alpha_2$. The broadness of multifractal spectrum is defined as $B = \alpha_2 - \alpha_1$, where $f(\alpha)$ is the spectrum function in terms of Holder regularity indices α's. d is usually taken as 0.2.