CNR-IMATI Milano

December 14-16, 2004

WAVELETS AND SELF-SIMILARITY: THEORY AND APPLICATIONS

Lecture 2: Bayesian Modeling with Wavelets

P L A N

1. **Bayes' Paradigm**

2. **Bayesian Modeling in Wavelet Domain**

3. **BAMS**

4. **Other Bayesian Wavelet Procedures and Their Use**

The Reverend

• The *model* for a typical observation X conditional on unknown parameter θ is $f(x|\theta)$. As a function of θ, $f(x|\theta) = \ell(\theta)$ is called *likelihood.* The functional form of f is fully specified up to parameter θ.

• The parameter θ is supported by the parameter space Θ and is considered random variable. The random variable θ has distribution $\pi(\theta)$ that is called *prior.*

- If the prior for θ is specified up to parameter τ, $\pi(\theta|\tau)$, τ is called *hyperparameter*.
- The distribution $h(x, \theta) = f(x|\theta)\pi(\theta)$ is called *joint* distribution.
- Joint distribution can be factorized differently, $h(x, \theta) = \pi(\theta|x)m(x)$.
- The distribution $\pi(\theta|x)$ is called *posterior* distribution for θ, given x.
- The *marginal* distribution $m(x)$ can be obtained by integrating out the parameter θ from the joint distribution $h(x, \theta)$,

$$m(x) = \int_\Theta h(x, \theta) d\theta = \int_\Theta f(x|\theta)\pi(\theta) d\theta.$$

• Therefore, the posterior $\pi(\theta|x)$ can be expressed as

$$\pi(\theta|x) = \frac{h(x,\theta)}{m(x)} = \frac{f(x|\theta)\pi(\theta)}{m(x)} = \frac{f(x|\theta)\pi(\theta)}{\int_\Theta f(x|\theta)\pi(\theta)d\theta}.$$

• Suppose $Y \sim f(y|\theta)$ is to be observed. (Posterior) predictive distribution of Y, given observed $X = x$ is

$$f(y|x) = \int_\Theta f(y|\theta)\pi(\theta|x)d\theta.$$

The marginal distribution $m(y) = \int_\Theta f(y|\theta)\pi(\theta)d\theta$ is sometimes called prior predictive distribution.

- Wavelet Transforms $\Longrightarrow$ Wavelet Domains
- STATISTICAL MODELING in Wavelet Domains instead in the domains of original data.
  - Critically Sampled (Orthogonal) [The topic of Lecture 1]
  - Redundant (Stationary, Continuous, Frames)
  - What is Better: Parsimony or Redundancy?

Example: Minimal and Redundant Wavelet Domains.

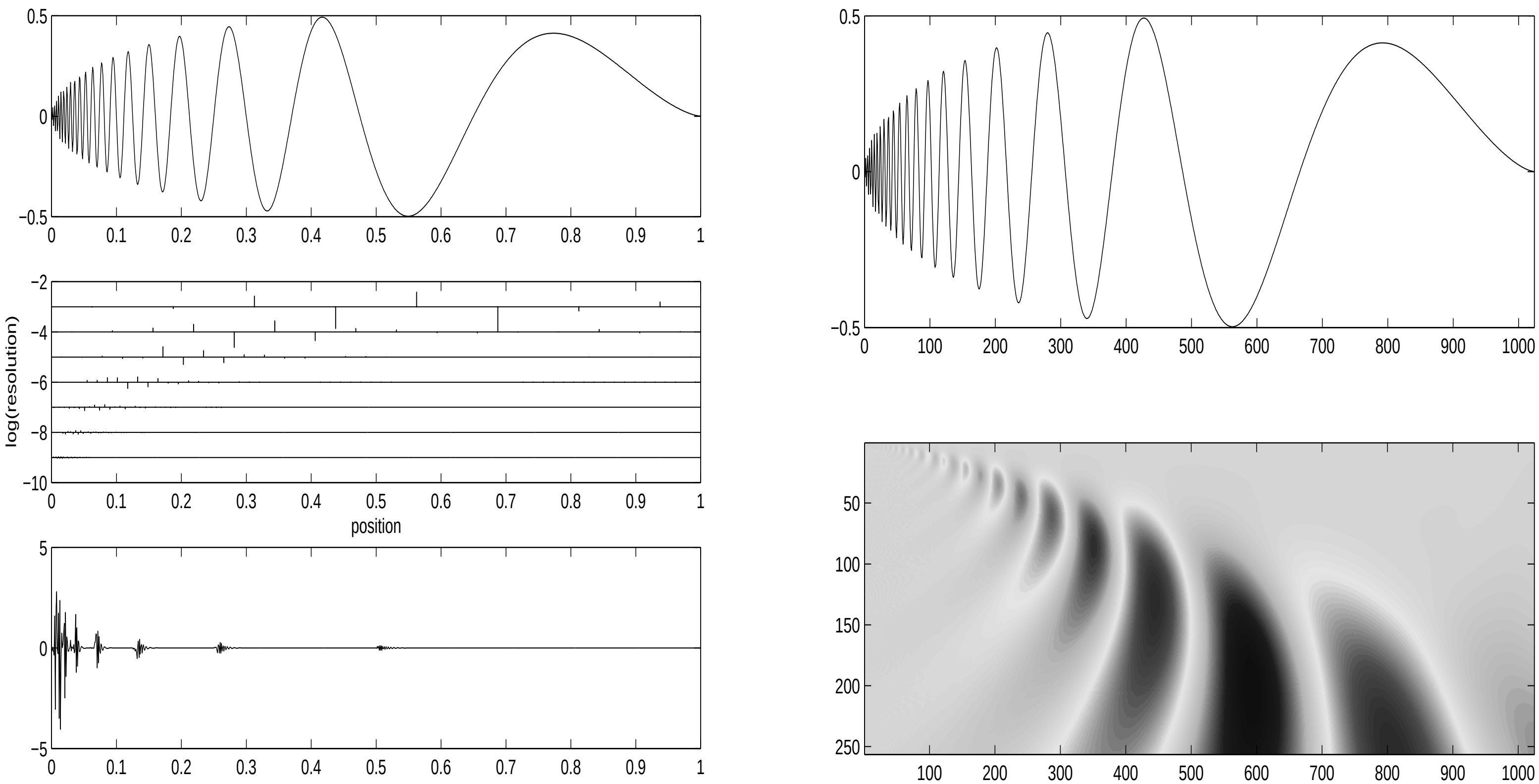


Figure 1: (left) DWT; (right) CWT of Doppler Signal

■ Wavelet-like and Related Transformations

■ Atomic Decompositions, Pursuits, General Time / Frequency Methods.

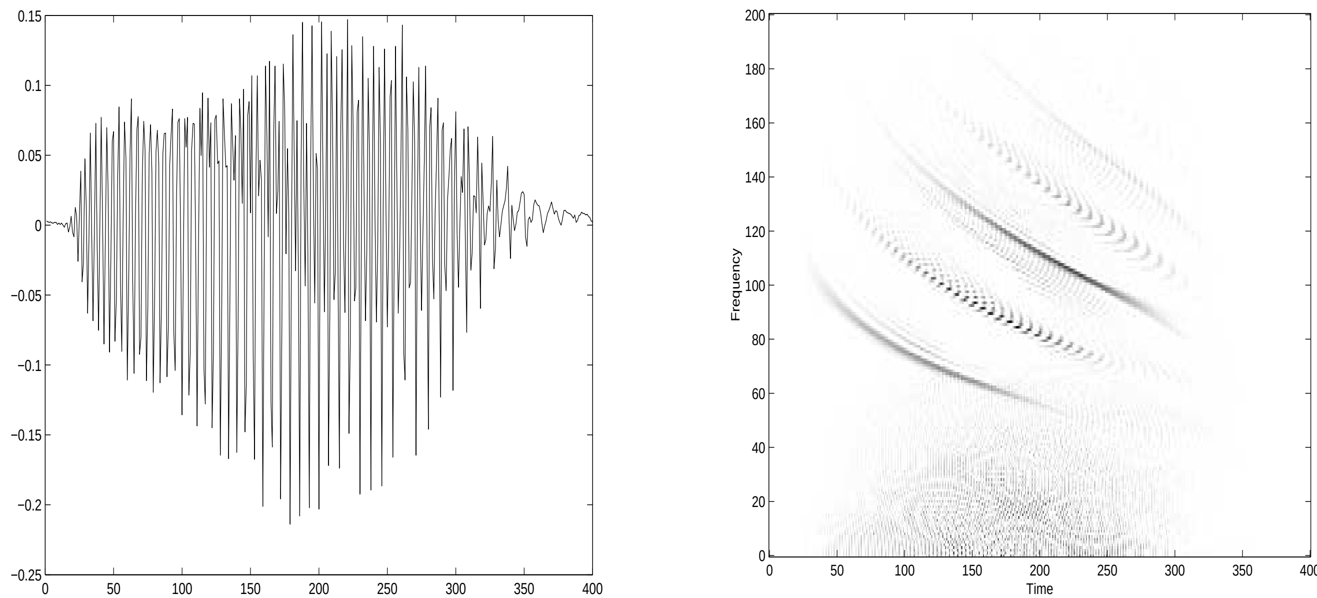


Figure 2: (left) Bat signal; (right) Its Wigner-Ville distribution

Model Based Wavelet Data Processing

$$\boxed{\textbf{DATA}} \xrightarrow{\mathbb{W}} \boxed{\textbf{Wavelet Coefficients: “Detail” \& “Smooth”}}$$

$$\|$$

$$\boxed{\textbf{Processed DATA}} \xleftarrow{\mathbb{W}^{-1}} \boxed{\textbf{Process (Detail) Coefficients}}$$

Process $\equiv$

- Shrink
- Transform
- Simulate New, Construct
- Resample
- Split

■ Various shrinkage/thresholding methods. (Start with Donoho, Johnstone,& Coauthors, the early 1990's)

■ Shrinkage induced by statistical modeling in the wavelet domain.

$$\boxed{\underset{\sim}{y} = \underset{\sim}{f} + \underset{\sim}{\epsilon} \qquad \xrightarrow{\mathbb{W}} \qquad \underset{\sim}{d} = \underset{\sim}{\theta} + \underset{\sim}{\epsilon}}$$

■ Estimate θ by $\hat{\theta}$ and set $\hat{f}$ as $\mathbb{W}^{-1}(\hat{\theta})$

Location model on d, $f(\mathbf{d} - \underset{\sim}{\theta}|\text{parameters})$

- Dimensionality of the model (Do not worry – wavelets decorrelate)
- Accounting for dependence (neighbors, parent-children), Blocking strategies (classical), Many Bayes solutions (MCMC, hidden MC's).
- **Model complexity/efficiency compromise.**
  - Simple models/Fast shrinkage ◦ Realistic?
  - Complex models ◦ Useful?

Our Focus: **Bayesianly Induced Shrinkage**

- Why Bayes? Prior information about regularity, self-similarity, periodicity, energy, and modality of the signal.
- $\hat{\theta}$ – Bayes Rule. Does it shrink? Well, not always.
- Prior Selection [Priors on θ, σ^2, hyper-priors]; What is our intuition about parameters in the wavelet-domain models?
- Need for Empirical Bayes.

- Complex Models: Often computationally prohibitive, Require MCMC and “MCMC-educated” users, Practitioners uneasy about.

- Simple Models: Efficient Shrinkage.

Normal Likelihood + Plug-in (hyper)Parameters $\rightarrow$ Observations of empirical wavelet coefficients disagree with the model!

Example: colorblue **BAMS** (**B**ayesian **A**daptive **M**ultiresolution **S**moother).

- A compromise between model reality/efficiency.

Model:

$$[d|\theta, \sigma^2] \sim \mathcal{N}(\theta, \sigma^2); \quad \sigma^2 \sim \mathcal{E}(\mu), \quad \mu > 0.$$

Marginal Model (σ^2 is out!):

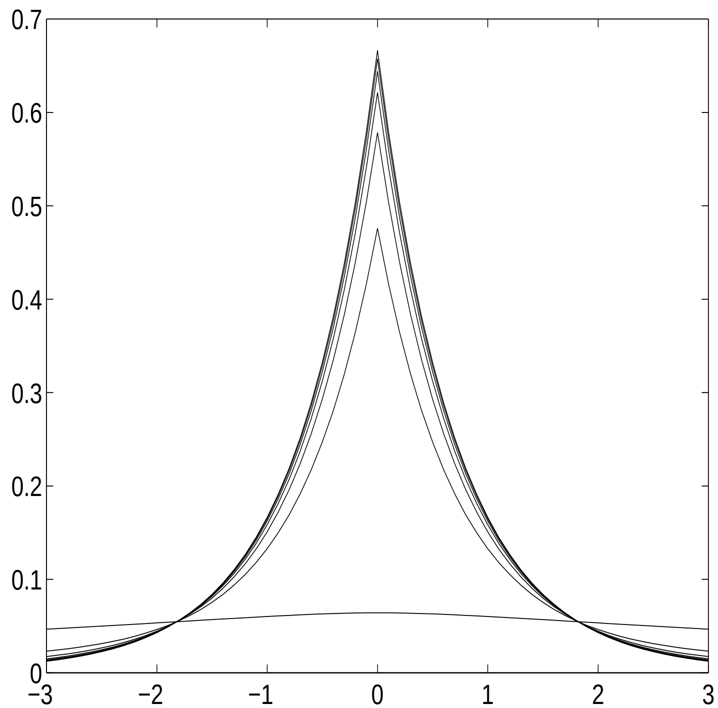
$$[d|\theta] \sim \mathcal{DE}(\theta, \frac{1}{\sqrt{2\mu}}); \quad [\frac{1}{2}\sqrt{2\mu}e^{-\sqrt{2\mu}|d-\theta|}].$$

Prior:

$$[\theta|\epsilon] \sim \epsilon\delta_0 + (1-\epsilon)\mathcal{DE}(0, \tau), \quad \epsilon = \epsilon(\text{multiresolution level}).$$

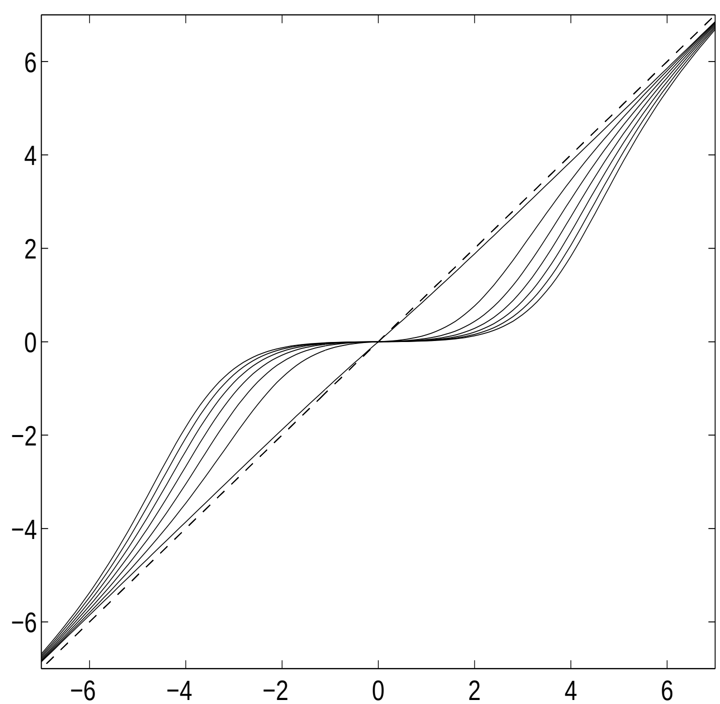
Marginal (Should agree with **d**):

$$d \sim m(d) = \epsilon \mathcal{DE}(0, \frac{1}{\sqrt{2\mu}}) + (1-\epsilon)\frac{\tau e^{-|d|/\tau} - \frac{1}{\sqrt{2\mu}} e^{-\sqrt{2\mu}|d|}}{2\tau^2 - 1/\mu}.$$



Bayes Rule:

$$\hat{\theta} = \delta(d) = \frac{\tau(\tau^2 - \sigma^2)de^{-d/\tau} + \tau^2/\mu(e^{-d\sqrt{2\mu}} - e^{-d/\tau})}{(\tau^2 - 1/(2\mu))(\tau e^{-d/\tau} - (1/\sqrt{2\mu})e^{-d\sqrt{2\mu}})}.$$



Specification of Hyperparameters:

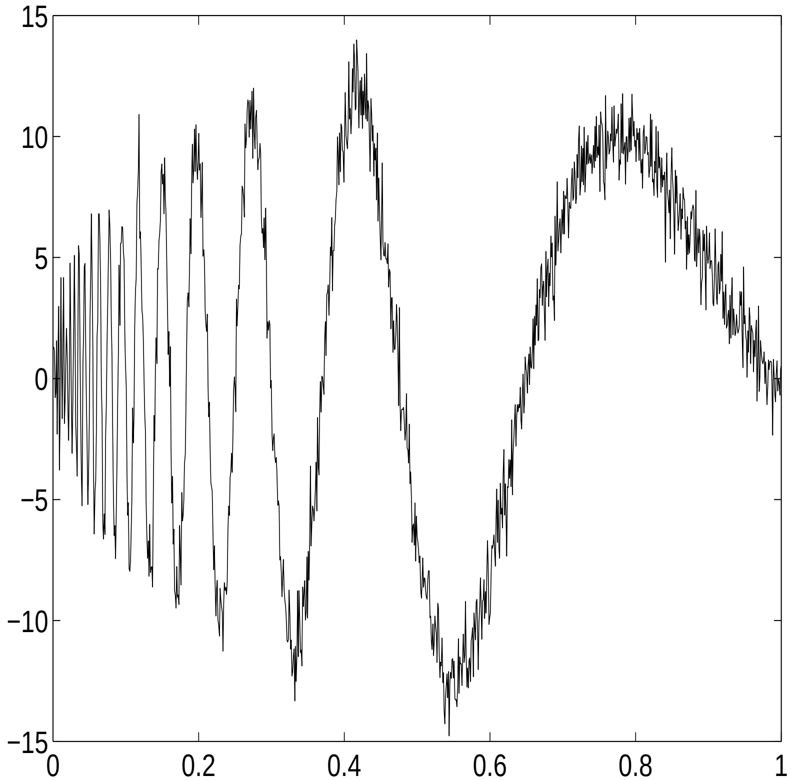
- $\mu = \frac{1}{\texttt{pseudos}}$

$\texttt{pseudos} = |Q_1 - Q_3|/C$, where Q_1 and Q_3 are the first and the third quartile of the finest level of details in the decomposition and $1.3 \leq C \leq 1.5$.

- $\epsilon(j) = 1 - \frac{1}{(j - \texttt{coarsest} + 1)^{\gamma}}, \ \gamma = \frac{3}{2}$.
- $\tau = \sqrt{\sigma_d^2 - \frac{1}{\mu}}$. (Information on selfsimilarity via τ.)

FUNCTION	VisuShrink	SureShrink
BLOCKS	0.6840 (0.0719 + 0.6122)	0.2225 (0.1369 + 0.0856)
BUMPS	1.5707 (0.1165 + 1.4543)	0.6827 (0.2660 + 0.4167)
DOPPLER	0.4850 (0.0523 + 0.4327)	0.2285 (0.0946 + 0.1340)
HEAVISINE	0.1204 (0.0339 + 0.0864)	0.0949 (0.0416 + 0.0534)

FUNCTION	ABWS (CKM Chicago '97)	BAMS
BLOCKS	0.0995 (0.0874 + 0.0121)	0.1107 (0.0965 + 0.0142)
BUMPS	0.3495 (0.2228 + 0.1267)	0.3404 (0.1976 + 0.1428)
DOPPLER	0.1646 (0.1006 + 0.0640)	0.1482 (0.0899 + 0.0584)
HEAVISINE	0.0874 (0.0442 + 0.0433)	0.0815 (0.0511 + 0.0304)



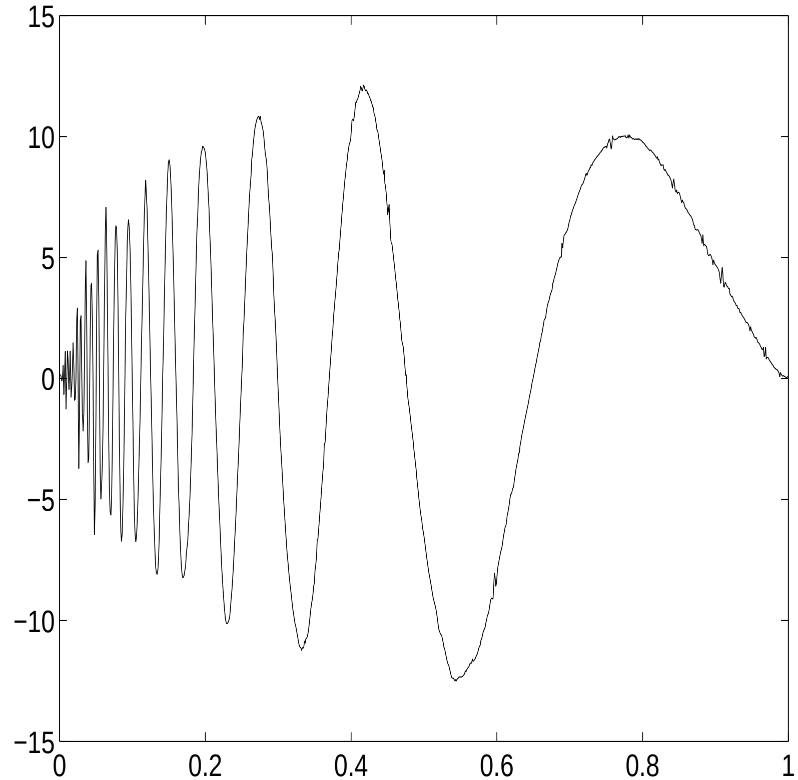


Figure 3: Heavisine signal: $n = 1024$, SNR=7.

Matlab program implementing BAMS:

`http://www.isye.gatech.edu/~brani/wavelet.html`

From: Ruggeri, F. and Vidakovic, B. (2005). Bayesian Modeling in the Wavelet Domain, To appear as Chapter in Handbook of Statistics Vol. 25, on Bayesian Statistics (C.R. Rao and Dipak Dey).

Some other Bayesian Procedures + Annotated Bibliography

Chipman, Kolaczyk, and McCulloch (1997) based on the stochastic search variable selection (SSVS)

$$[d|\theta] \sim \mathcal{N}(\theta, \sigma^2).$$

$$[\theta|\gamma_j] \sim \gamma_j \mathcal{N}(0, (c_j\tau_j)^2) + (1-\gamma_j)\mathcal{N}(0, \tau_j^2),$$

where

$$[\gamma_j] \sim Ber(p_j).$$

$$\delta(d) = \left[P(\gamma_j = 1|d) \frac{(c_j \tau_j)^2}{\sigma^2 + (c_j \tau_j)^2} + P(\gamma_j = 0|d) \frac{\tau_j^2}{\sigma^2 + \tau_j^2} \right] \, d,$$

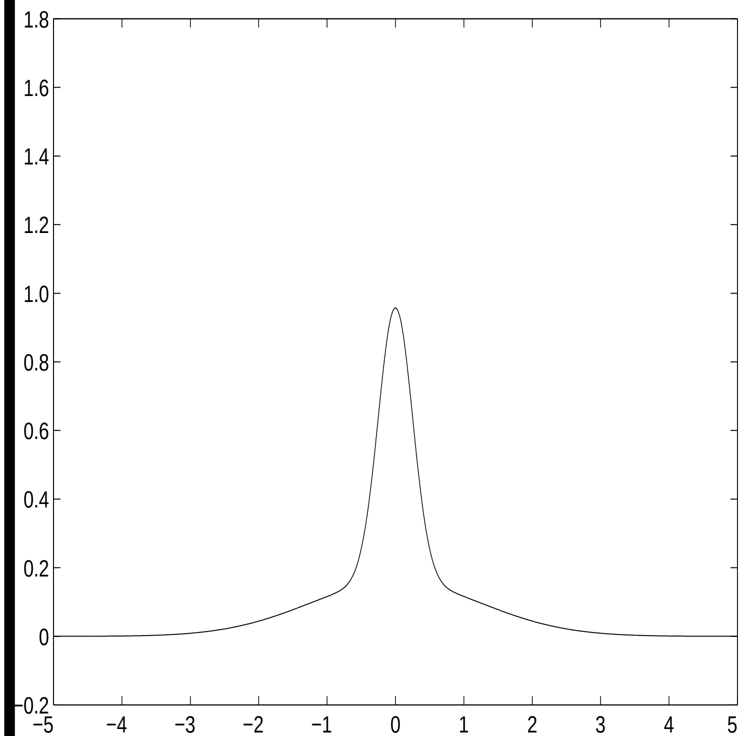
where

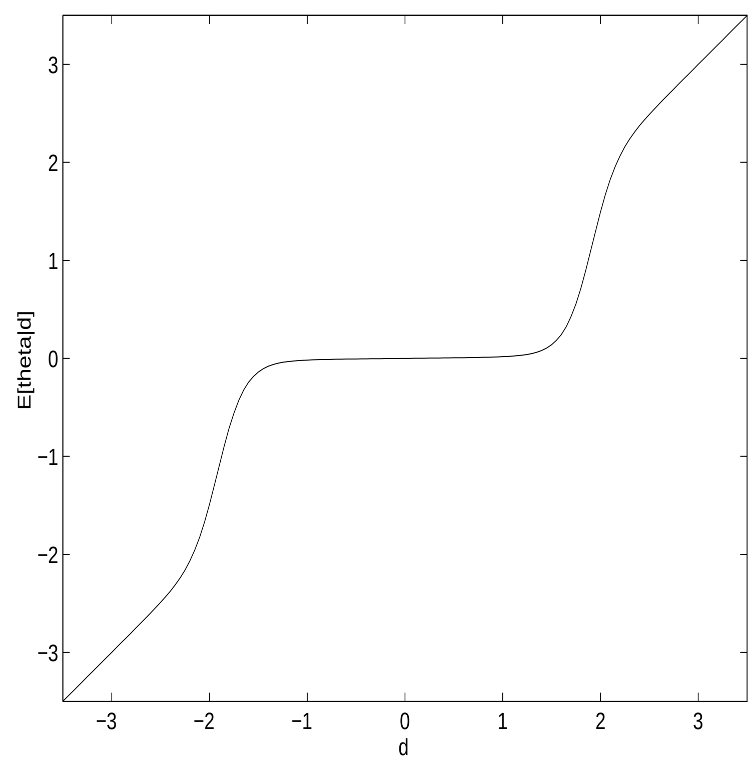
$$P(\gamma_j = 1|d) = \frac{p_j \pi(d|\gamma_j = 1)}{(1 - p_j)\pi(d|\gamma_j = 0)}$$

and

$$\pi(d|\gamma_j = 1) = \phi_{\sigma^2 + (c_j \tau_j)^2}(d) \;\; \text{and} \;\; \pi(d|\gamma_j = 0)\phi_{\sigma^2 + \tau_j^2}(d).$$

De Canditiis and Vidakovic (2004) extend the ABWS method to multivariate case (block shrinkage) and unknown σ^2 using a mixture of normal-inverse Gamma priors.


1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
-0.2
−5
−4
−3
−2
−1
0
1
2
3
4
5


E[theta|d]
3
2
1
0
−1
−2
−3
d

Clyde, Parmigiani, and Vidakovic (1998): *Biometrika*

$$\begin{aligned}[\theta|\gamma_j, \sigma^2] &\sim \mathcal{N}(0, (1-\gamma_j) + \gamma_j c_j \sigma^2), \\ \left[\lambda\nu/\sigma^2\right] &\sim \chi^2_\nu,\end{aligned}$$

where λ and ν are fixed hyperparameters.

$$\gamma_j \sim \mathcal{B}er(p_j)$$

The posterior mean of $\boldsymbol{\theta}|\boldsymbol{\gamma}$ is

$$E(\boldsymbol{\theta}|\boldsymbol{d}, \boldsymbol{\gamma}) = \Gamma(I_n + C^{-1})^{-1}\boldsymbol{d},$$

where Γ and C are diagonal matrices with γ_{jk} and c_{jk}, respectively, on the diagonal and 0 elsewhere.

The posterior mean is obtained by averaging over all models. Model averaging leads to a multiple shrinkage estimator of θ:

$$E(\boldsymbol{\theta}|\boldsymbol{d}) = \sum_{\boldsymbol{\gamma}} \pi(\boldsymbol{\gamma}|\boldsymbol{d})\Gamma\left(I_n + C^{-1}\right)^{-1}\boldsymbol{d},$$

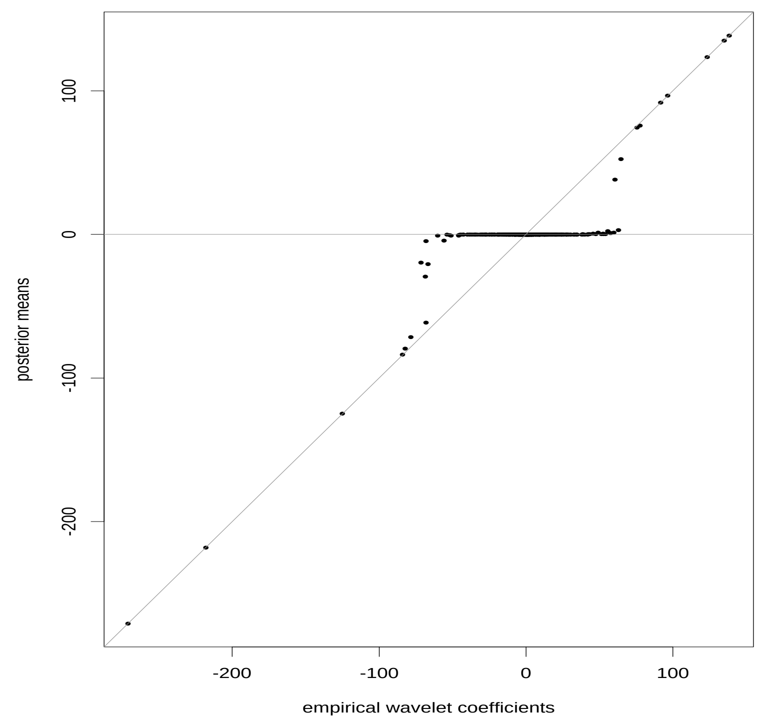
where $\pi(\boldsymbol{\gamma}|\boldsymbol{d})$ is the posterior probability of a particular subset $\boldsymbol{\gamma}$.

The approximation can be achieved by either conditioning on σ (plug-in approach) or by assuming independence of the elements in $\boldsymbol{\gamma}$.

$$\begin{aligned}
\pi(\boldsymbol{\gamma}|\boldsymbol{d}) &\approx \prod_{j,k} \rho_{jk}^{\gamma_{jk}}(1-\rho_{jk})^{1-\gamma_{jk}} \\
\rho_{jk}(\boldsymbol{d},\sigma) &= \frac{a_{jk}(\boldsymbol{d},\sigma)}{1+a_{jk}(\boldsymbol{d},\sigma)},
\end{aligned}$$

where

$$\begin{aligned}
a_{jk}(\boldsymbol{d},\sigma) &= \frac{p_{jk}}{1-p_{jk}}(1+c_{jk})^{-1/2}\cdot\exp\left\{\frac{1}{2}\frac{S_{jk}^2}{\sigma^2}\right\} \\
S_{jk}^2 &= d_{jk}^2/(1+c_{jk}^{-1}).
\end{aligned}$$



posterior means
100
0
-100
-200
-200
-100
0
100
empirical wavelet coefficients

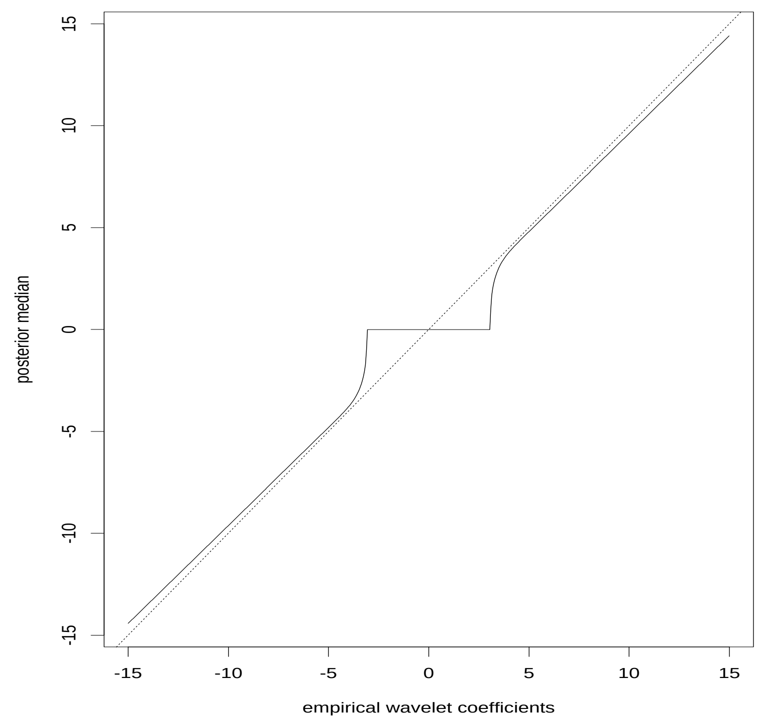
Abramovich, Sapatinas, and Silverman (1998) use weighted absolute error loss and show that for a prior on θ

$$[\theta] \sim \pi_j \mathcal{N}(0, \tau_j^2) + (1 - \pi_j)\delta(0)$$

and normal $\mathcal{N}(\theta, \sigma^2)$ likelihood, the posterior median is

$$Med(\theta|d) = \text{sign}(d) \max(0, \zeta). \tag{1}$$

$$\begin{aligned} \zeta &= \frac{\tau_j^2}{\sigma^2 + \tau_j^2}|d| - \frac{\tau_j \sigma}{\sqrt{\sigma^2 + \tau_j^2}} \Phi^{-1}\left(\frac{1 + \min(\omega, 1)}{2}\right), \text{ and} \\ \omega &= \frac{1 - \pi_j}{\pi_j} \frac{\sqrt{\tau_j^2 + \sigma^2}}{\sigma} \exp\left\{-\frac{\tau_j^2 d^2}{2\sigma^2(\tau_j^2 + \sigma^2)}\right\}. \end{aligned}$$


posterior median
empirical wavelet coefficients
-15
-10
-5
0
5
10
15

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