CNR-IMATI Milano
December 14, 2004

WAVELETS AND SELF-SIMILARITY: THEORY AND APPLICATIONS

Lecture 1: Basics of Wavelets

P L A N

1. **What are Wavelets**
2. **Why Wavelets**
3. **Mathematics of Wavelets**
4. **Discrete Wavelet Transform**
5. **2-D Case and Wavelet Packets**

What are wavelets?

Jean Baptiste Joseph Fourier 1768-1830 and Alfred Haar 1885-1933

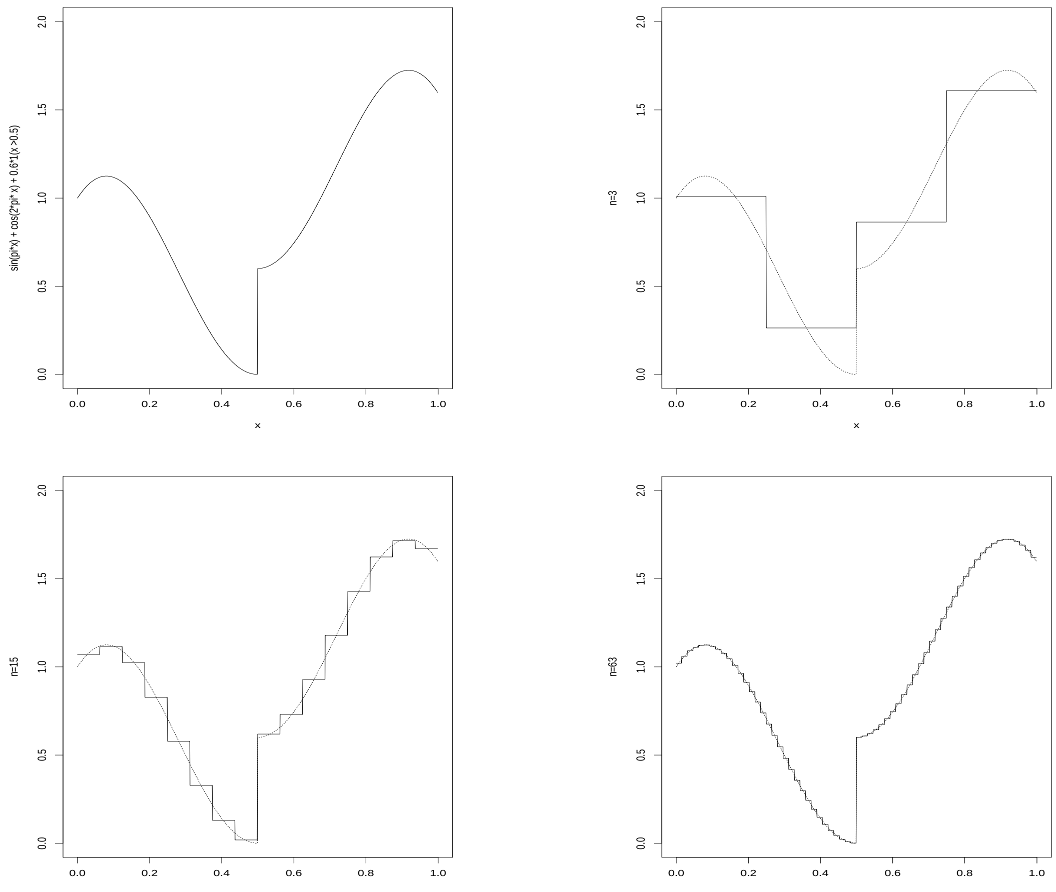
The first "wavelet basis" was discovered in 1910 when Alfred Haar showed that any continuous function $f(x)$ on $[0, 1]$ can be approximated by

$$f_n(x) = \langle \xi_0, f \rangle \xi_0(x) + \langle \xi_1, f \rangle \xi_1(x) + \cdots + \langle \xi_n, f \rangle \xi_n(x).$$

The Haar basis is very simple:

$$\begin{aligned}
\xi_0(x) &= \mathbf{1}(0 \le x \le 1), \\
\xi_1(x) &= \mathbf{1}(0 \le x \le 1/2) - \mathbf{1}(1/2 \le x \le 1), \\
\xi_2(x) &= \sqrt{2}[\mathbf{1}(0 \le x \le 1/4) - \mathbf{1}(1/4 \le x \le 1/2)], \\
&\cdots \\
\xi_n(x) &= 2^{j/2}[\mathbf{1}(k \cdot 2^{-j} \le x \le (k+1/2) \cdot 2^{-j}) \\
&\quad -\mathbf{1}((k+1/2) \cdot 2^{-j} \le x \le (k+1) \cdot 2^{-j})], \ldots
\end{aligned}$$

where n is decomposed as $n = 2^j + k,\ j \ge 0,\ 0 \le k \le 2^j - 1.$

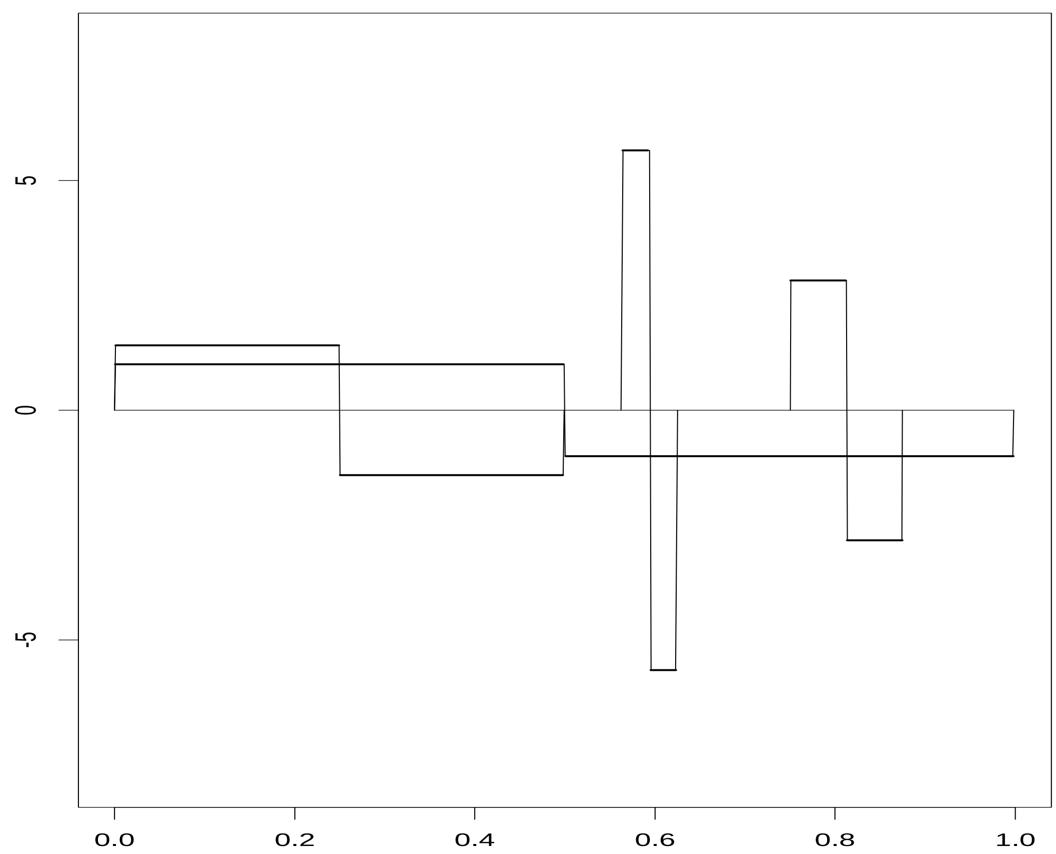

sin(pi*x) + cos(2*pi* x) + 0.6*1(x >0.5)
n=3
n=15
n=63
x
0.0
0.2
0.4
0.6
0.8
1.0
0.5
1.5
2.0

Function
$f(x) = \sin \pi x + \cos 2\pi x + 0.6 \cdot \mathbf{1}(x > 1/2),\ 0 \le x \le 1,$
and three different levels of approximation in the Haar basis. Approximations f_3, f_{15}, and f_{63} are plotted.

For any $n > 1$ the basis function ξ_n can be expressed as a scale-shift transform of a single function ξ_1,

$$\xi_n(x) = \xi_{j,k}(x) = 2^{j/2}\xi_1(2^j \cdot x - k), \quad n = 2^j + k.$$

- $\xi_n,\ n \ge 1$ describe the details.
- $\xi_0(x)$ is responsible for the "average."



Functions ξ_1, ξ_2, ξ_{14}, and ξ_{25} from the Haar basis on $[0, 1]$.

- The Haar wavelet decomposition of the function

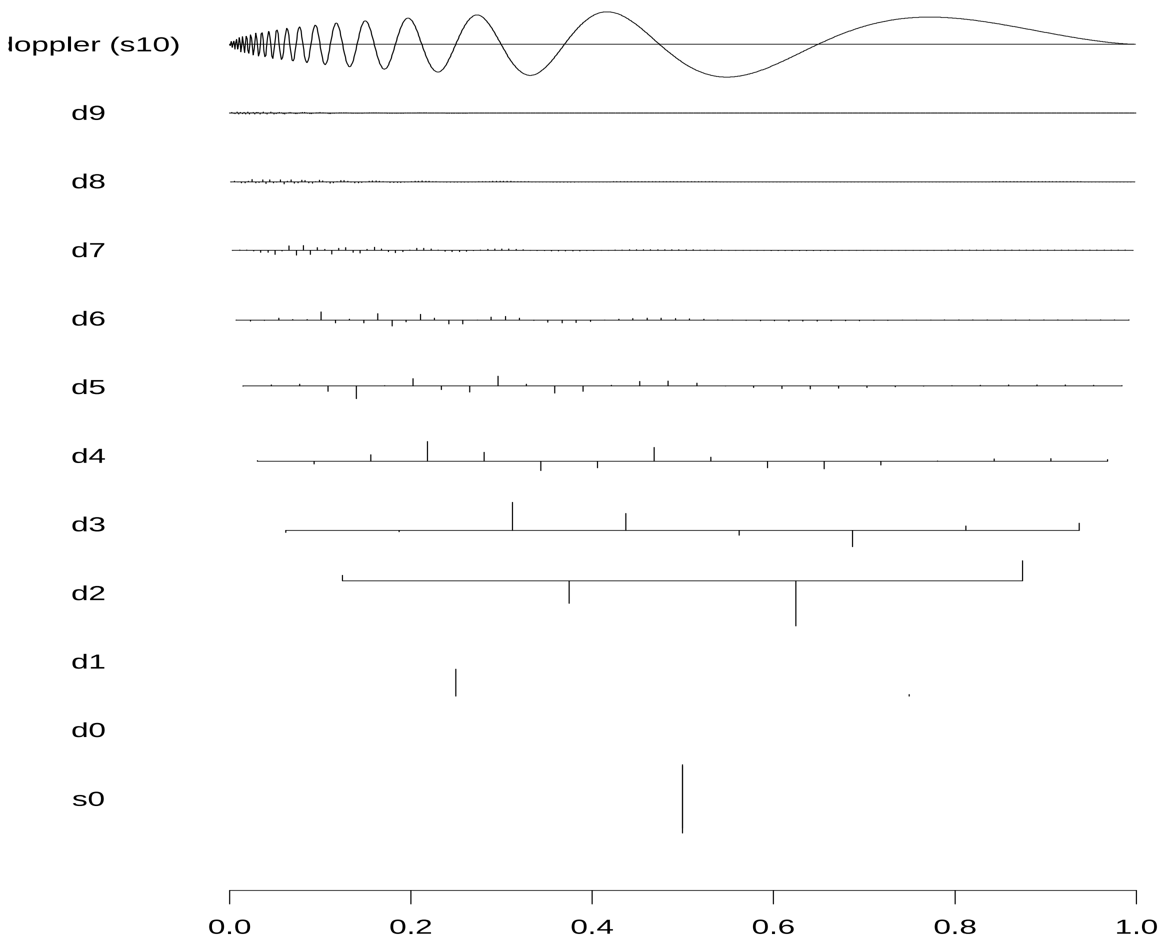
$$y(x) = \sqrt{x(1-x)}\ \sin\frac{2.1\pi}{x+0.05},\ \ 0 \le x \le 1.$$

- The function is known as the `doppler` function (Donoho and Johnstone)

- Implemented as a test function in almost all wavelet software packages.

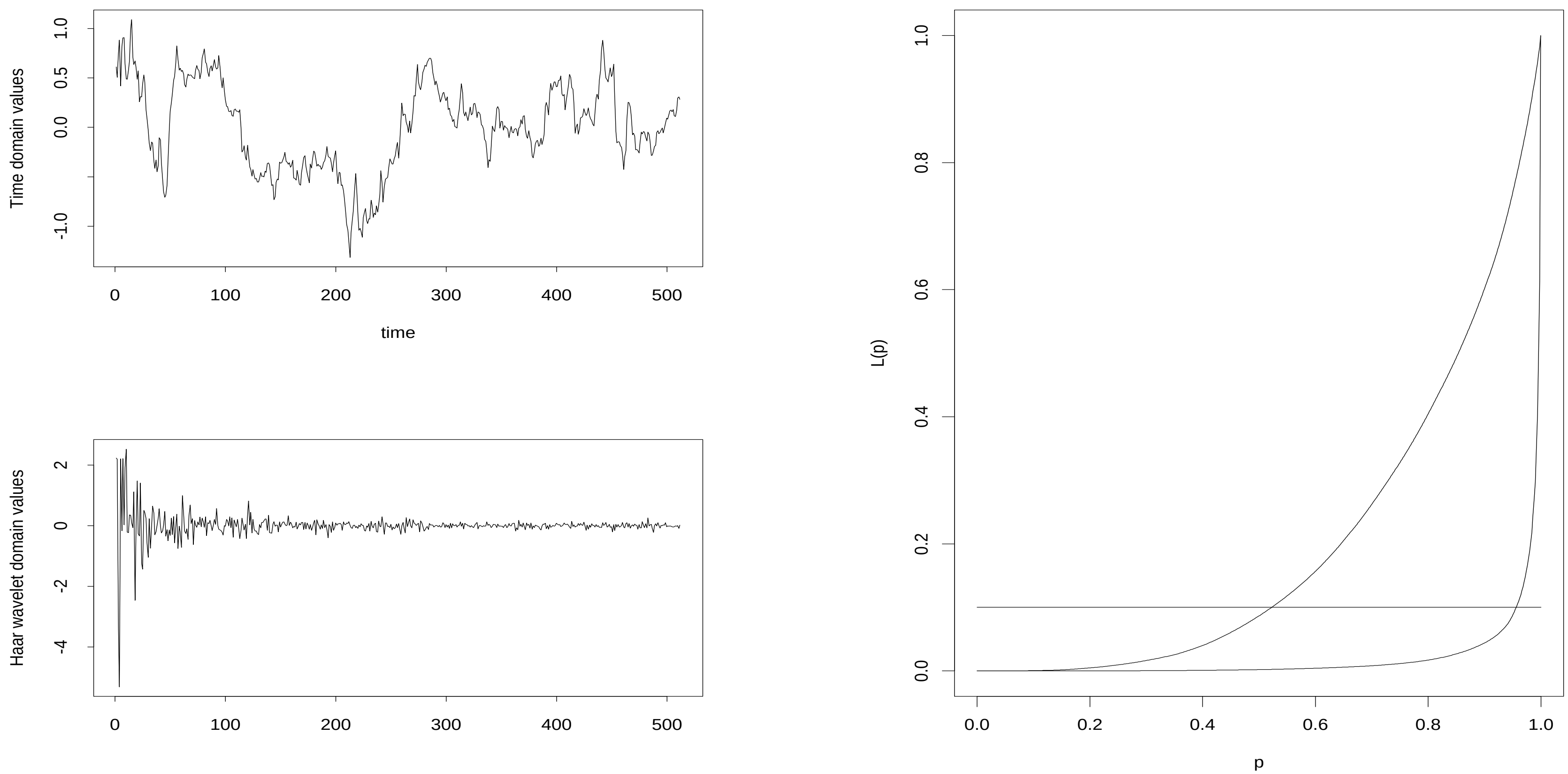
level	coef			j and k :	$n = 2^j + k$
		coefficients of	support	j	k
c0	$\boldsymbol{c}$	ξ_0	1		
d0	$\boldsymbol{d}_0$	ξ_1	1	$j = 0$	$k = 0$
d1	$\boldsymbol{d}_1$	ξ_2 - ξ_3	$1/2$	$j = 1$	$0 \leq k \leq 1$
d2	$\boldsymbol{d}_2$	ξ_4 - ξ_7	$1/4$	$j = 2$	$0 \leq k \leq 3$
d3	$\boldsymbol{d}_3$	ξ_8 - ξ_{15}	$1/8$	$j = 3$	$0 \leq k \leq 2^3 - 1$
		...	...	...	...
d9	$\boldsymbol{d}_9$	ξ_{512} - ξ_{1023}	$1/2^9$	$j = 9$	$0 \leq k \leq 2^9 - 1$

Coefficients of `doppler` function in Haar basis presented in levels determined by the length of support of corresponding basis functions, $\xi_n, \; n \geq 0$.

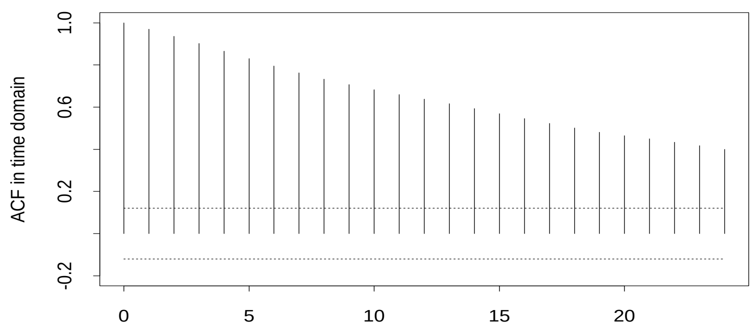


loppler (s10)
d9
d8
d7
d6
d5
d4
d3
d2
d1
d0
s0
0.0
0.2
0.4
0.6
0.8
1.0

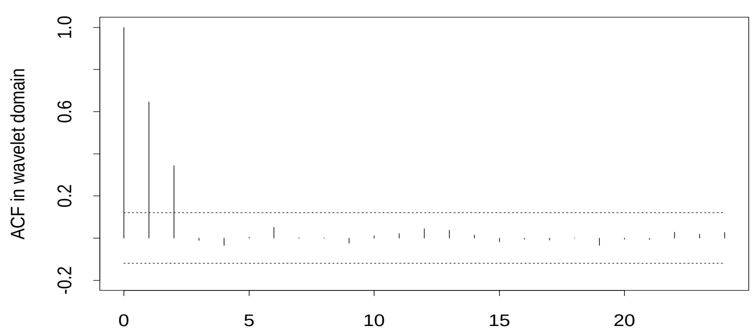
- **Wavelets disbalance the data.**



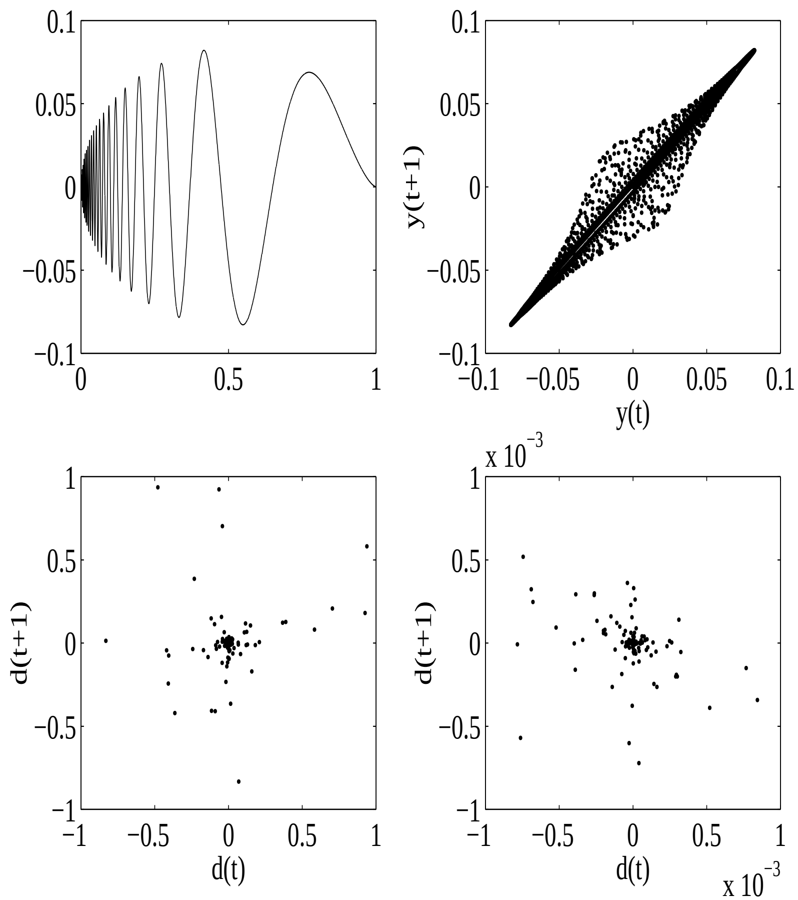
• Wavelets whiten the data.



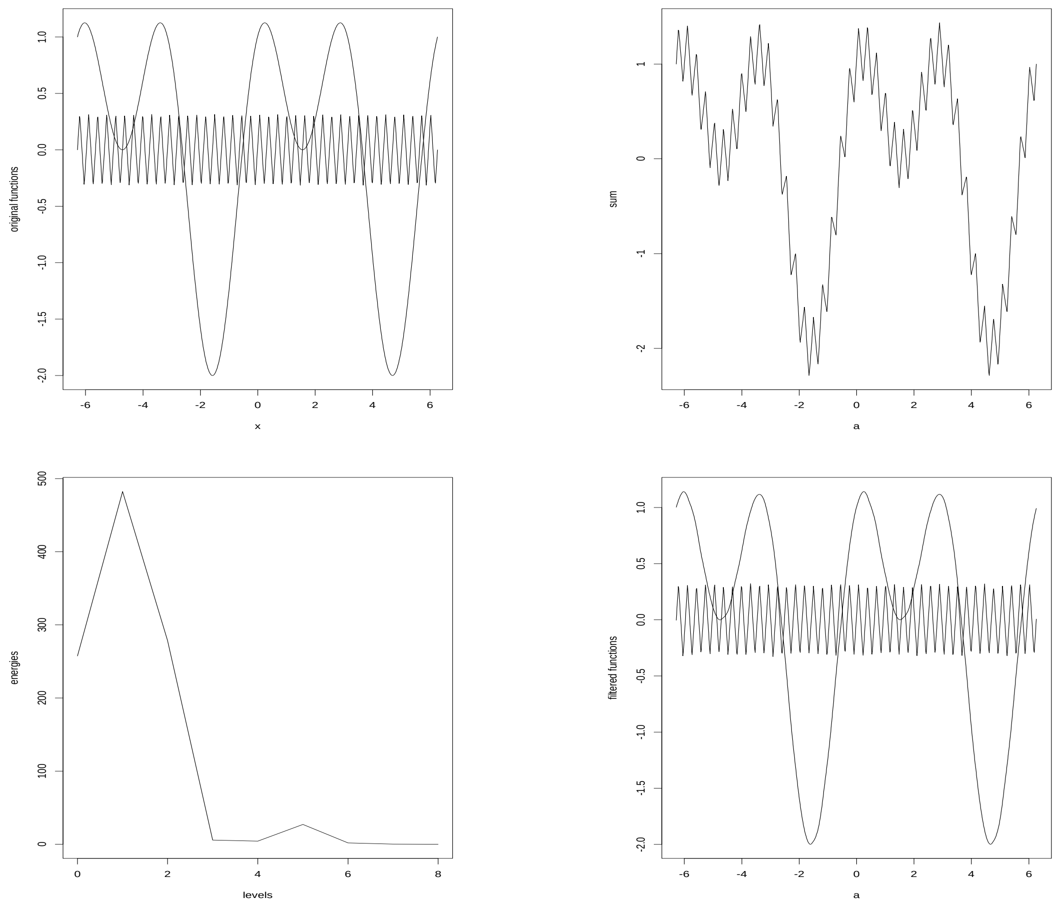
(a)



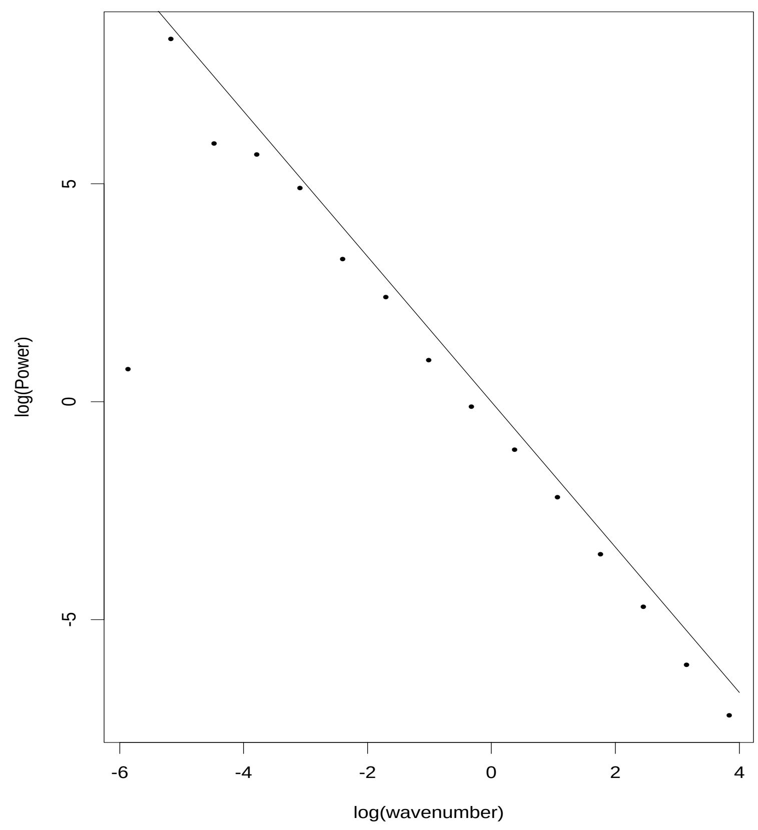
(b)



- **Wavelets filter the data.**

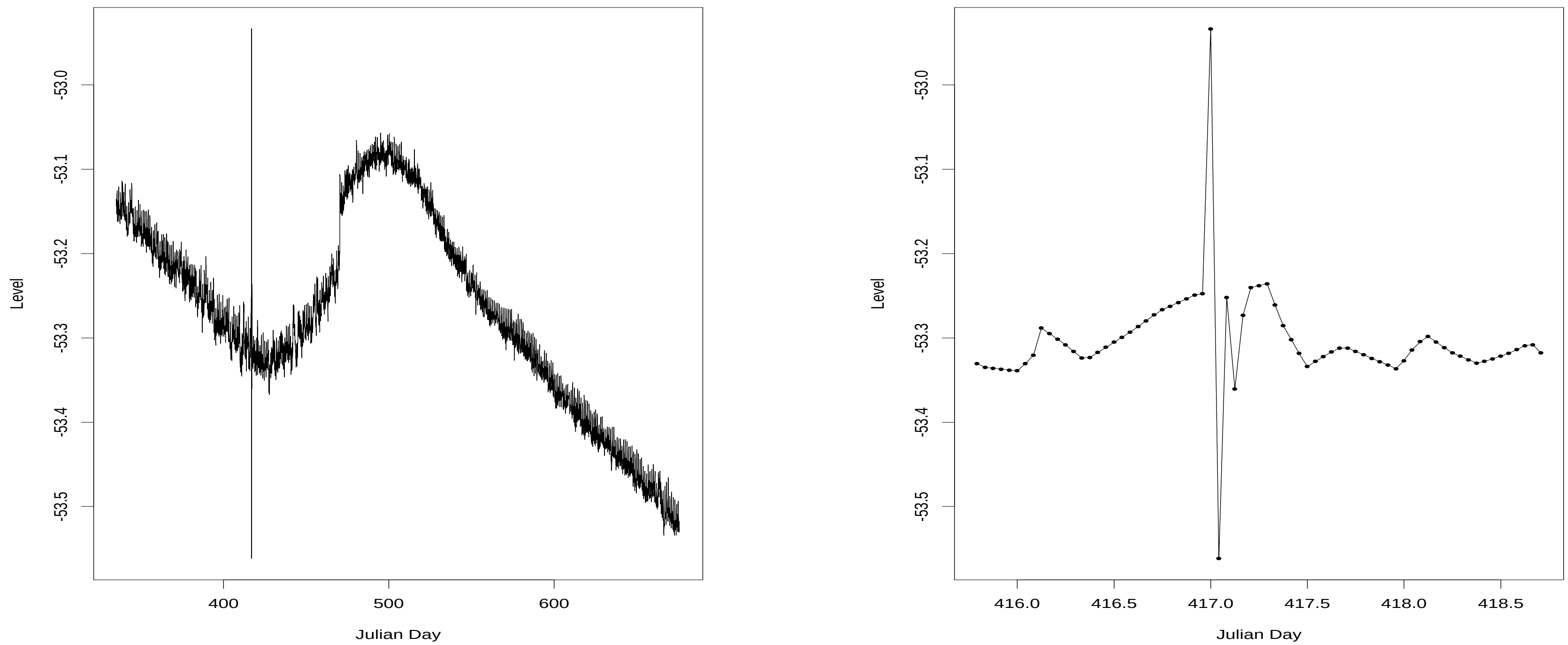


- **Wavelets detect self-similarity in data.**

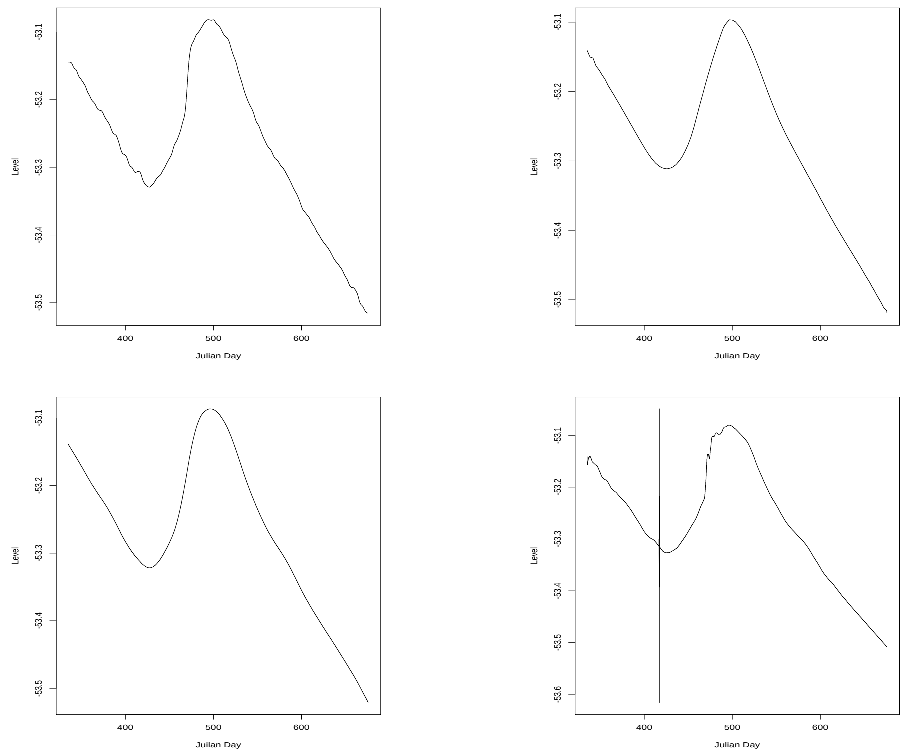


California Earthquakes

- A researcher from the Geology Department at Duke University was interested in the possibility of predicting earthquakes by monitoring water levels.
- Water level measurements from six wells located in California, were taken every hour for approximately six years.
- The goal was to smooth the data, eliminate the noise, and inspect the signal at pre-earthquake time.



Raw data for hourly measurements (one year, $8192 = 2^{13}$ observations). The line-like artifact (enlarged in right panel) corresponds to the earthquake time (Julian day of 417).



Comparison of several smoothing methods. Upper, Left: Data smoothed by kernel method (normal window, k=5); Upper, Right: Data smoothed by `loess` method; Lower, Left: Data smoothed by `supsmu` method; Lower, Right: Wavelet Smoothed Data

<u>Multiresolution analysis</u> (MRA) is a sequence of closed subspaces $V_n, n \in Z$ in $L_2(R)$

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots .$$

$$\cap_n V_n = \{\mathbf{0}\}, \quad \overline{\cup_n V_n} = L_2(R).$$

The hierarchy in MRA is constructed such that

- V-spaces are self similar,

$$f(x) \in V_0 \text{ iff } f(2^n x) \in V_n.$$

• There exists *a scaling function* $\phi \in V_0$ whose integer translates span V_0,

$$V_0 = \{f \in L_2(R)| \ \ f(x) = \sum_k c_k \phi(x-k)\},$$

and for which the set $\{\phi(\bullet - k), \ k \in Z\}$ is an orthonormal basis.

• (Scaling equation) Since $V_0 \subset V_1$,

$$\boxed{\phi(x) = \sum_{k \in Z} h_k \ \sqrt{2}\phi(2x-k).}$$

- **Normalization.**

$$\boxed{\sum_{k\in Z} h_k = \sqrt{2}.}$$

- **Orthogonality.** For any $l \in Z$,

$$\boxed{\sum_k h_k h_{k-2l} = \delta_l.}$$

An important special case is $l = 0$

$$\boxed{\sum_k h_k^2 = 1.}$$

Link between h and ϕ

Transfer function:

$$m_0(\omega) = \frac{1}{\sqrt{2}} \sum_{k \in Z} h_k e^{-ik\omega} \quad [= \frac{1}{\sqrt{2}} H(\omega)].$$

Recall the scaling equation:

$$\boxed{\phi(x) = \sum_{k \in Z} h_k \sqrt{2} \phi(2x - k).}$$

In the Fourier domain:

$$\boxed{\Phi(\omega) = m_0\left(\frac{\omega}{2}\right) \Phi\left(\frac{\omega}{2}\right)}$$

$$\boxed{\Phi(\omega) = m_0\left(\tfrac{\omega}{2}\right)\Phi\left(\tfrac{\omega}{2}\right)}$$

$$\begin{aligned}
\Phi(\omega) &= \int_{-\infty}^{\infty} \phi(x) e^{-i\omega x} dx \\
&= \sum_k \sqrt{2}\; h_k \int_{-\infty}^{\infty} \phi(2x-k) e^{-i\omega x} dx \\
&= \sum_k \frac{h_k}{\sqrt{2}} e^{-ik\omega/2} \quad \int_{-\infty}^{\infty} \phi(2x-k) e^{-i(2x-k)\omega/2} d(2x-k) \\
&= \sum_k \frac{h_k}{\sqrt{2}} e^{-ik\omega/2} \;\; \Phi\left(\frac{\omega}{2}\right) \\
&= m_0\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right) \qquad \Longrightarrow \qquad \boxed{\Phi(\omega) = \textstyle\prod_{n=1}^{\infty} m_0\left(\frac{\omega}{2^n}\right).}
\end{aligned}$$

Mother Wavelet ψ

• Whenever sequence of subspaces satisfy MRA properties there exists an orthonormal basis for $L_2(R)$,

$$\{\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k),\ j, k \in Z\}$$

such that $\{\psi_{jk}(x),\ j\text{-fixed},\ k \in Z\}$ is an orthonormal basis for the "difference space" $W_j = V_{j+1} \ominus V_j$.

(because of containment $W_0 \subset V_1$),

$$\psi(x) = \sum_{k \in Z} g_k \sqrt{2}\phi(2x - k),$$

- Quadrature mirror relation

$$\boxed{g_n = (-1)^n \; h_{1-n}.}$$

- Any $L_2(R)$ function can be represented as

$$f(x) = \sum_{j,k} d_{jk}\psi_{jk}(x), \qquad [L_2(R) = \bigoplus_{j=-\infty}^{\infty} W_j .]$$

For any fixed j_0 the decomposition $L_2(R) = V_{j_0} \oplus \oplus_{j=j_0}^{\infty} W_j$ corresponds to representation

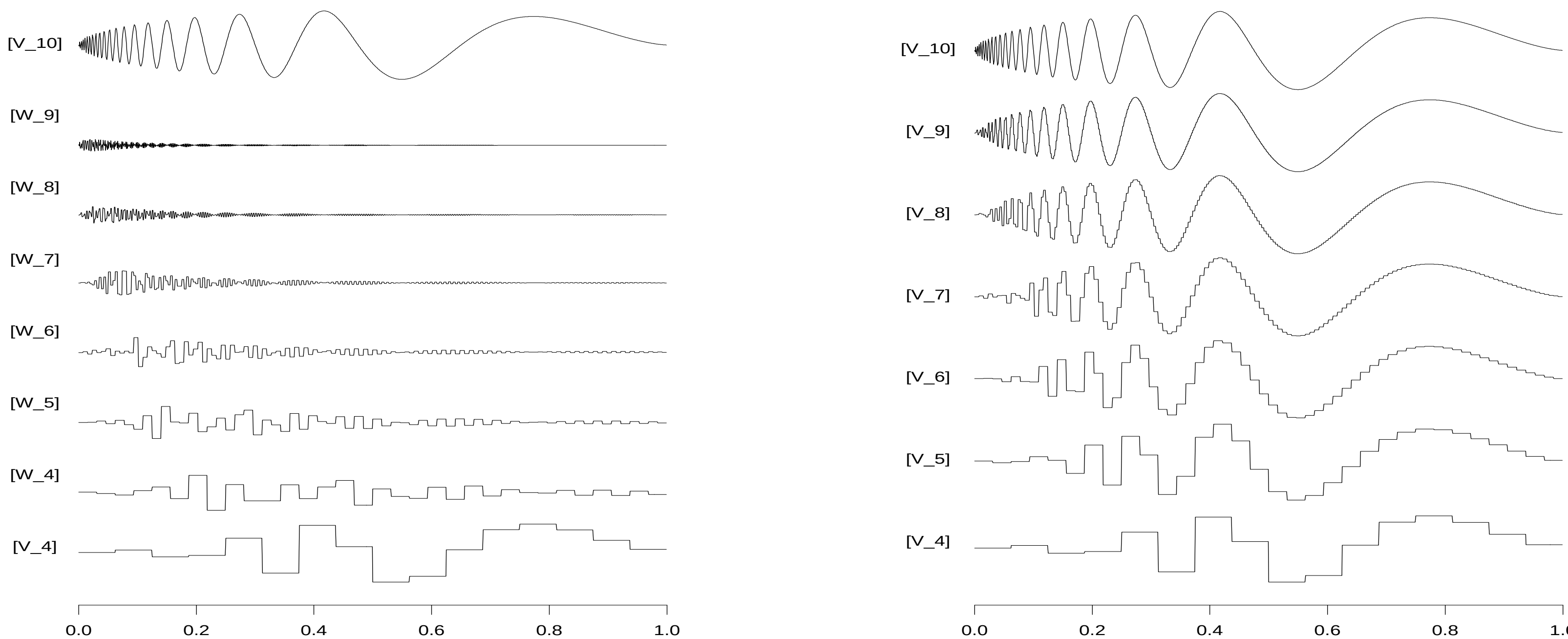
$$f(x) = \sum_k c_{j_0,k}\phi_{j_0,k}(x) + \sum_{j \geq j_0} \sum_k d_{jk}\psi_{j,k}(x).$$

Haar Example

$$\begin{aligned}\phi(x) &= \phi(2x) + \phi(2x-1) = \boxed{\tfrac{1}{\sqrt{2}}}\sqrt{2}\phi(2x) + \boxed{\tfrac{1}{\sqrt{2}}}\sqrt{2}\phi(2x-1), \\ \psi(x) &= \phi(2x) - \phi(2x-1) = \boxed{\tfrac{1}{\sqrt{2}}}\sqrt{2}\phi(2x) + \boxed{-\tfrac{1}{\sqrt{2}}}\sqrt{2}\phi(2x-1),\end{aligned}$$

which gives the wavelet filter coefficients:

$$\begin{aligned}h_0 &= h_1 = \frac{1}{\sqrt{2}}. \text{ and} \\ g_0 &= -g_1 = \frac{1}{\sqrt{2}}.\end{aligned}$$



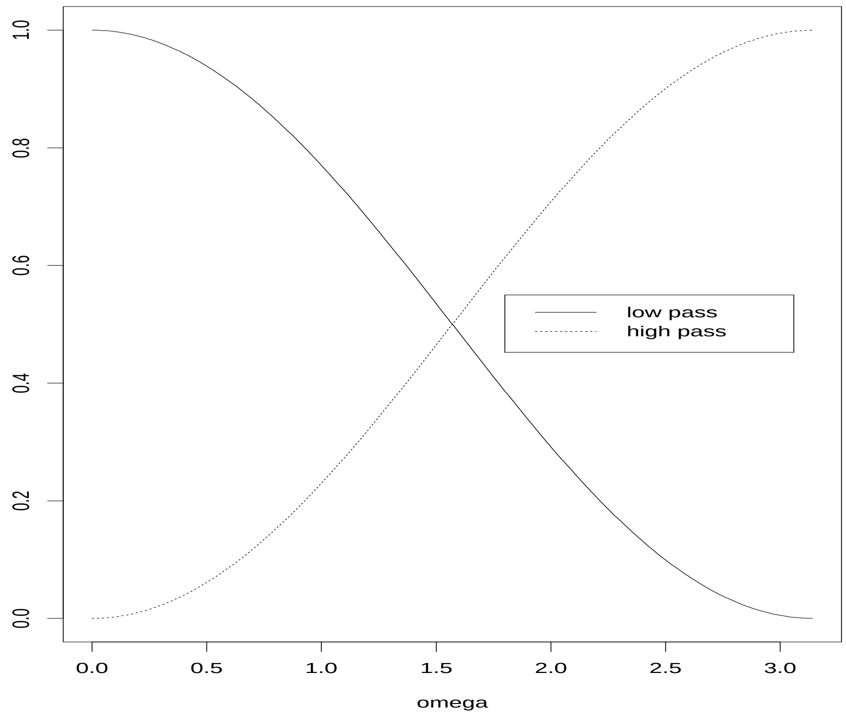
Left: Multiresolution Analysis of **doppler** function. The original function in V_{10} is a sum of projections on V_4 and W_4–W_9 subspaces. Right: Coarsening **doppler** function by projecting it to V-subspaces.

$$h_0 = h_1 = \frac{1}{\sqrt{2}}.$$

For the Haar wavelet, the transfer function becomes

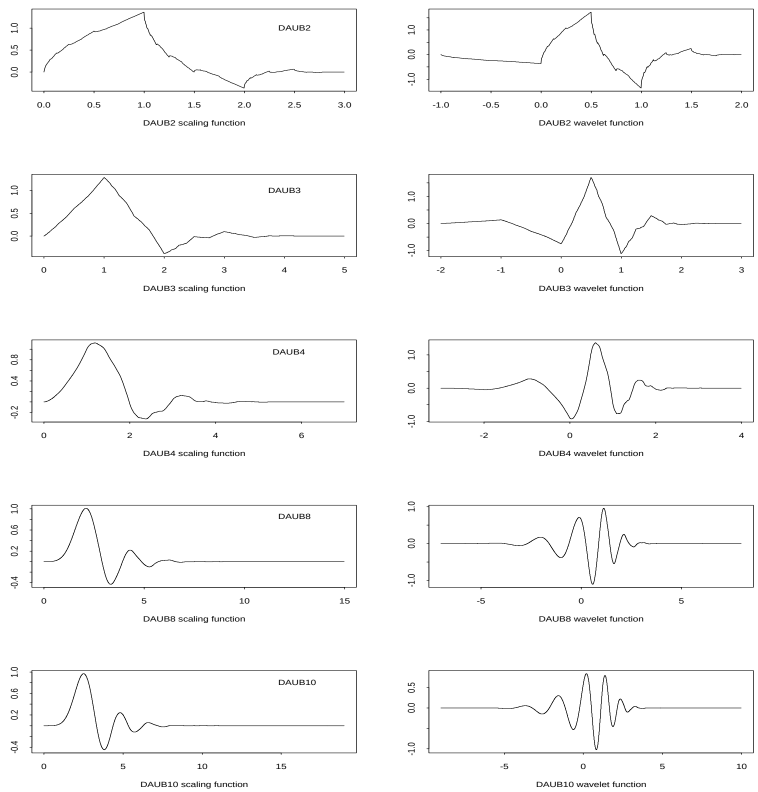
$$m_0(\omega) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}e^{-i\omega 0}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}e^{-i\omega 1}\right) = \frac{1+e^{-i\omega}}{2}.$$

$$m_1(\omega) = -e^{-i\omega}\ \overline{m_0(\omega+\pi)} = -e^{-i\omega}\left(\frac{1}{2} - \frac{1}{2}e^{i\omega}\right) = \frac{1-e^{-i\omega}}{2}.$$



Functions $|m_0(\omega)|^2 = \cos^2 \frac{\omega}{2}$ and $|m_1(\omega)|^2 = \sin^2 \frac{\omega}{2}$ for the Haar case.

Ingrid Daubechies was first to construct compactly supported orthogonal wavelets with preassigned degree of smoothness.



Graphs of scaling and wavelets functions from Daubechies family, $N = 2, 3, 4, 8,$ and 10.

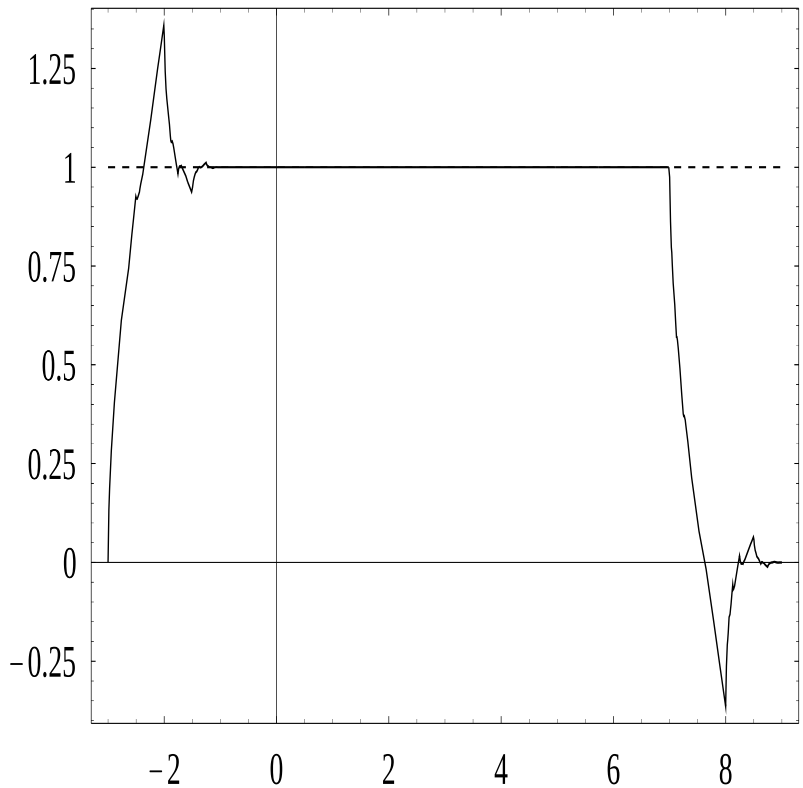
k	DAUB2	DAUB3	DAUB4	DAUB5
0	0.4829629131445	0.3326705529500	0.2303778133088	0.1601023979741
1	0.8365163037378	0.8068915093110	0.7148465705529	0.6038292697971
2	0.2241438680420	0.4598775021184	0.6308807679298	0.7243085284377
3	-0.1294095225512	-0.1350110200102	-0.0279837694168	0.1384281459013
4		-0.0854412738820	-0.1870348117190	-0.2422948870663
5		0.0352262918857	0.0308413818355	-0.0322448695846
6			0.0328830116668	0.0775714938400
7			-0.0105974017850	-0.00624149021273
8				-0.0125807519990
9				0.0033357252854

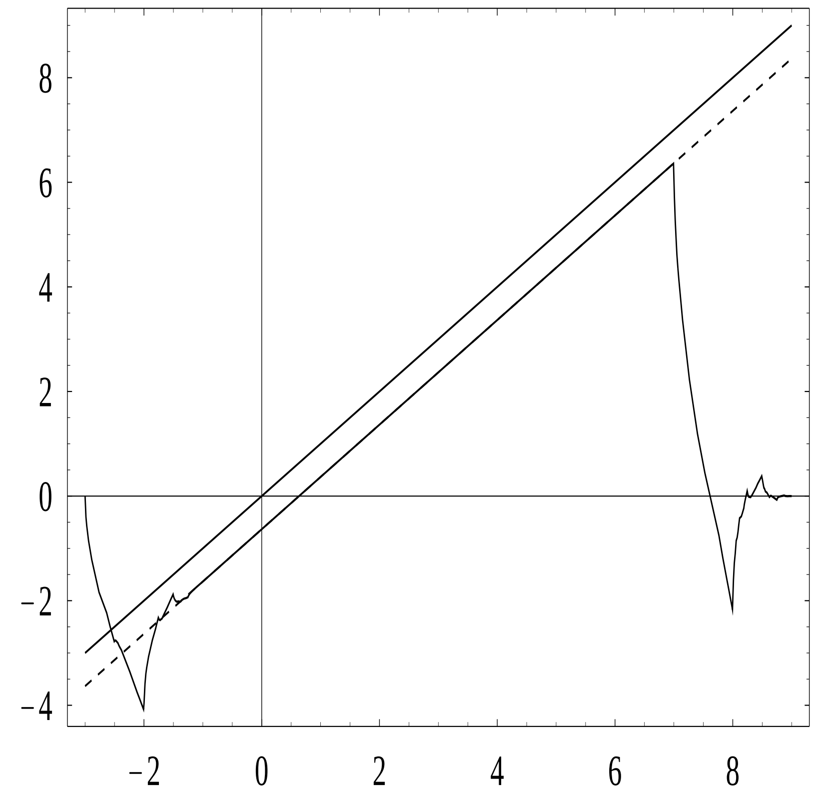
$$\begin{cases} h_0 + h_1 + h_2 + h_3 = \sqrt{2} \\ h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1 \\ -h_1 + 2h_2 - 3h_3 = 0 \\ h_0\ h_2 + h_1\ h_3 = 0 \end{cases}$$

- Pollen-type Parameterization [$s = 2\sqrt{2}$]:

n	h_n for $N = 2$
0	$(1 + \cos\varphi - \sin\varphi)/s$
1	$(1 + \cos\varphi + \sin\varphi)/s$
2	$(1 - \cos\varphi + \sin\varphi)/s$
3	$(1 - \cos\varphi - \sin\varphi)/s$

- Strang - Fix condition $[\sum_k \phi(x-k) = 1,$
$\sum_k (x - \text{Const})\phi(x-k) = x]$, ...



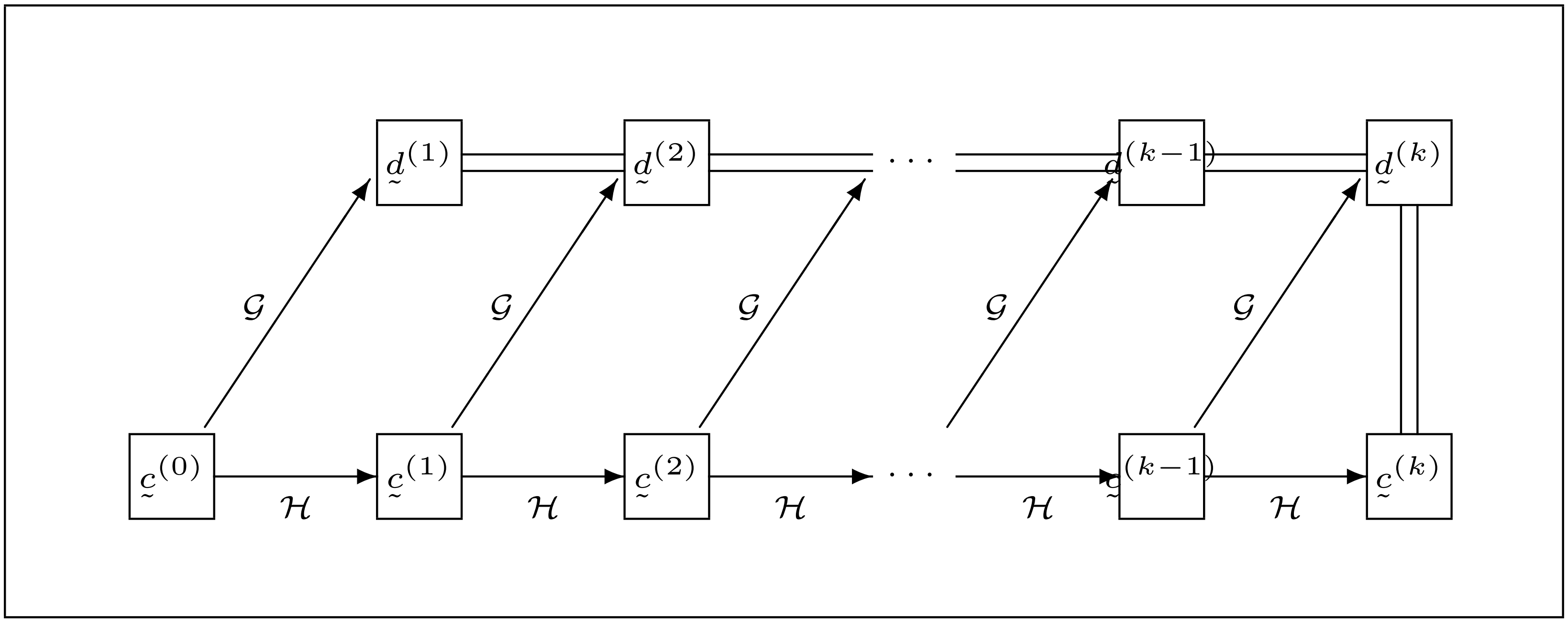


Discrete Wavelet Transforms

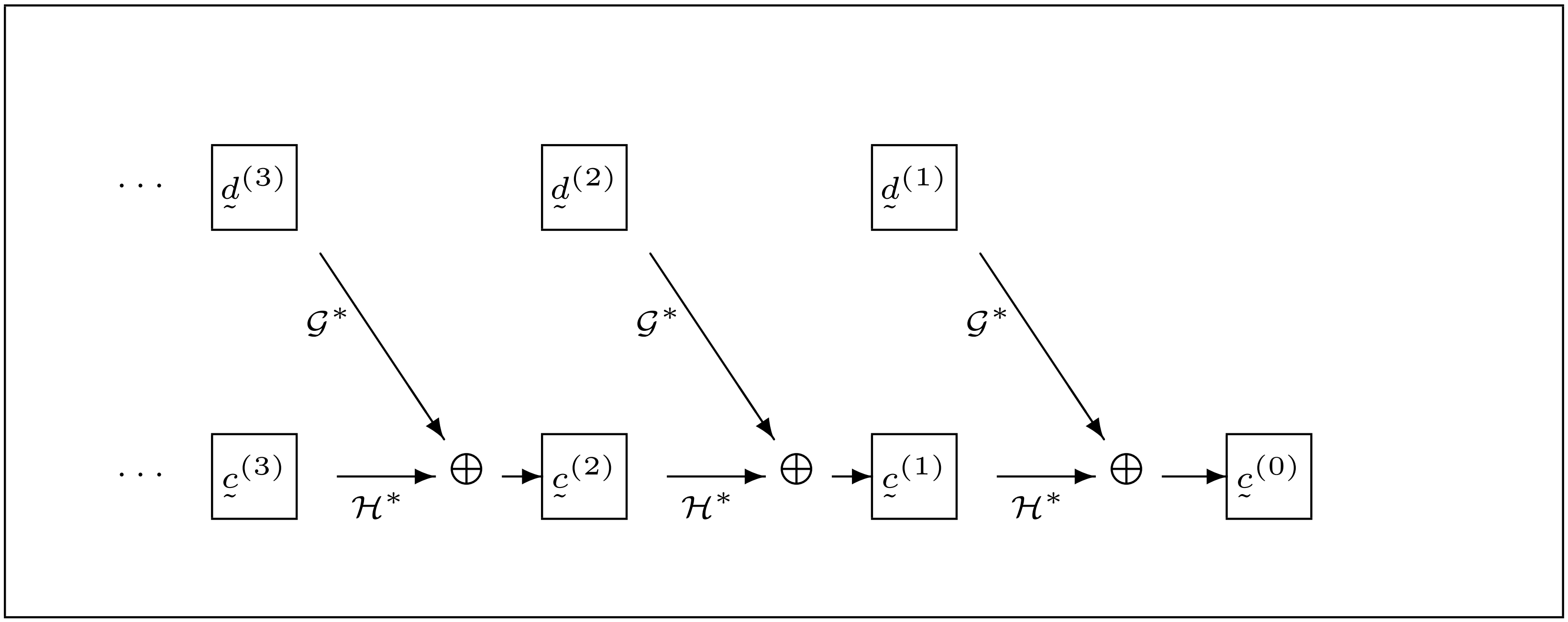
Fourier Methods	Fourier Integrals	Fourier Series	Discrete Fourier Transforms
Wavelet Methods	Continuous Wavelet Transforms	Wavelet Series	Discrete Wavelet Transforms

- **The Cascade Algorithm [Mallat, 1989]**

$$
\begin{aligned}
\mathcal{H}: \quad c_{j+1,l} &= \sum_k h_{k-2l} c_{j,k} \\
\mathcal{G}: \quad d_{j+1,l} &= \sum_k g_{k-2l} c_{j,k} \\
\mathcal{H}^*, \mathcal{G}^*: \quad c_{j,k} &= \sum_l c_{j+1,l} h_{k-2l} + \sum_l d_{j+1,l} g_{k-2l}.
\end{aligned}
$$

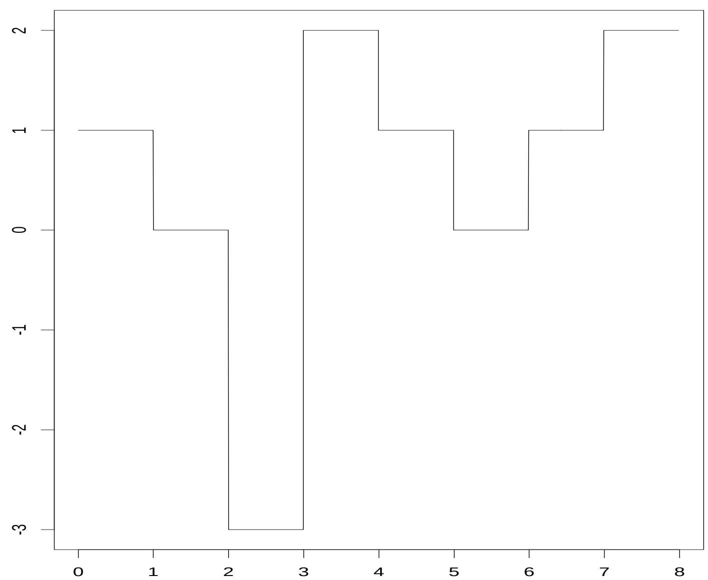


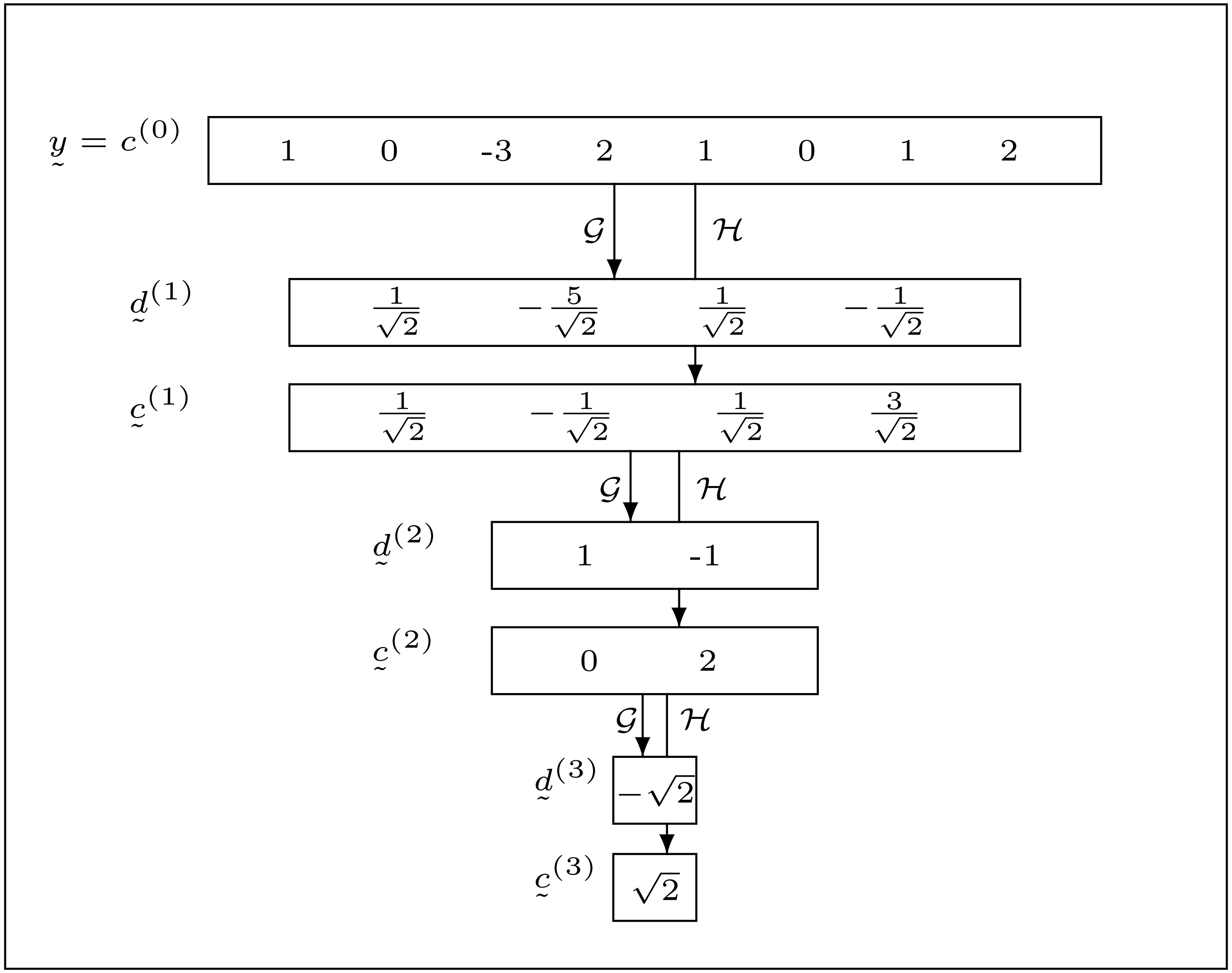
$\underset{\sim}{d}^{(1)}$
$\underset{\sim}{d}^{(2)}$
$\underset{\sim}{d}^{(k-1)}$
$\underset{\sim}{d}^{(k)}$
$\mathcal{G}$
$\underset{\sim}{c}^{(0)}$
$\underset{\sim}{c}^{(1)}$
$\underset{\sim}{c}^{(2)}$
$\underset{\sim}{c}^{(k-1)}$
$\underset{\sim}{c}^{(k)}$
$\mathcal{H}$

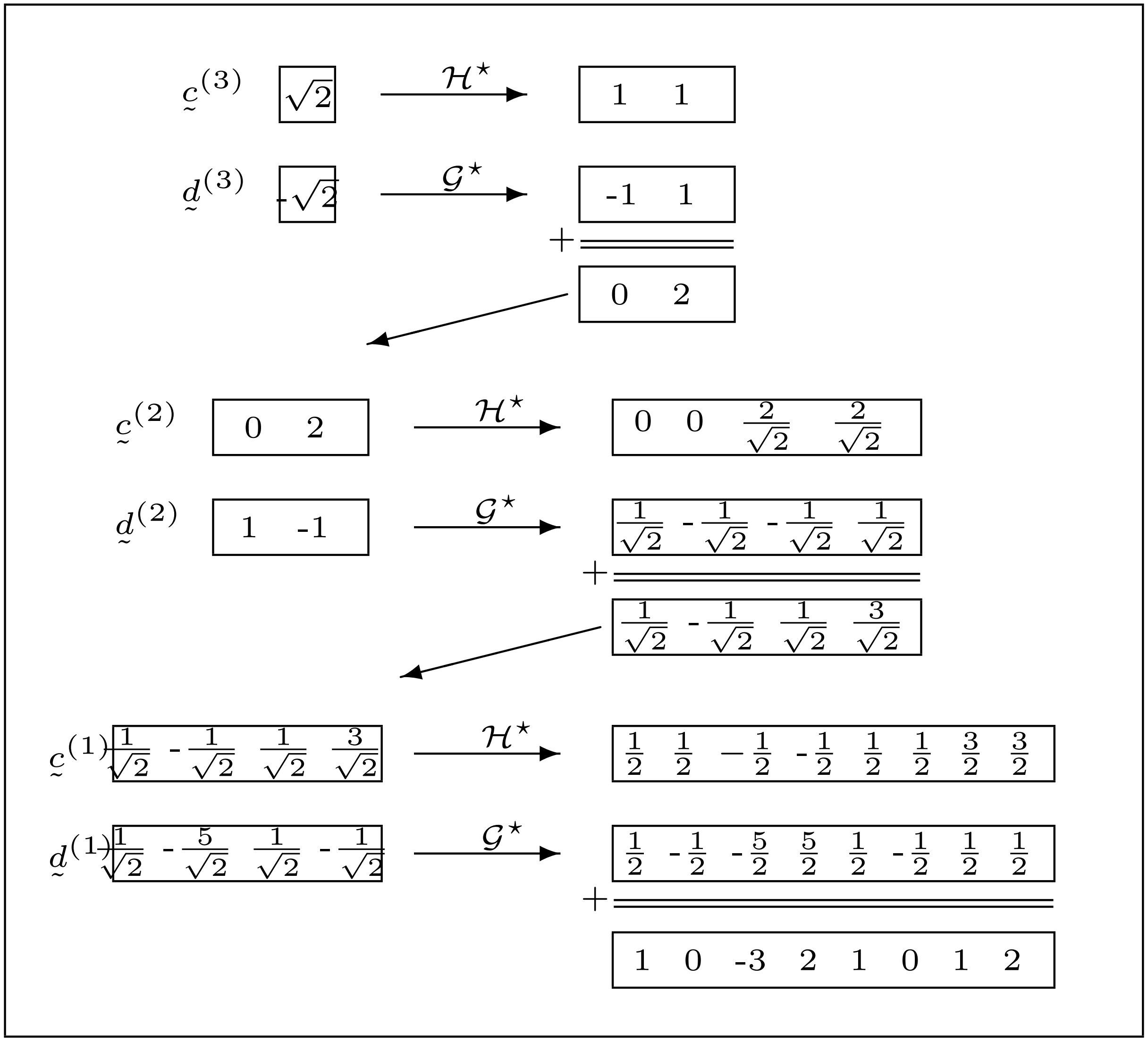

. . .
$\underset{\sim}{d}^{(3)}$
$\underset{\sim}{d}^{(2)}$
$\underset{\sim}{d}^{(1)}$
$\mathcal{G}^*$
$\mathcal{G}^*$
$\mathcal{G}^*$
. . .
$\underset{\sim}{c}^{(3)}$
$\mathcal{H}^*$
$\oplus$
$\underset{\sim}{c}^{(2)}$
$\mathcal{H}^*$
$\oplus$
$\underset{\sim}{c}^{(1)}$
$\mathcal{H}^*$
$\oplus$
$\underset{\sim}{c}^{(0)}$

Let $\underset{\sim}{y} = (1, 0, -3, 2, 1, 0, 1, 2)$. The values $f(n) = y_n,\ n = 0, 1, \ldots, 7$ are interpolated by piecewise constant function. Assume that f belongs to Haar's multiresolution space V_0.

$\underset{\sim}{h} = \{h_0, h_1\} = \{1/\sqrt{2}, 1/\sqrt{2}\}$; $\underset{\sim}{g} = \{g_0, g_1\} = \{1/\sqrt{2}, -1/\sqrt{2}\}$.




$\underset{\sim}{y} = c^{(0)}$
1 0 -3 2 1 0 1 2
$\mathcal{G}$ $\mathcal{H}$
$\underset{\sim}{d}^{(1)}$
$\frac{1}{\sqrt{2}}$ $-\frac{5}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ $-\frac{1}{\sqrt{2}}$
$\underset{\sim}{c}^{(1)}$
$\frac{1}{\sqrt{2}}$ $-\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ $\frac{3}{\sqrt{2}}$
$\mathcal{G}$ $\mathcal{H}$
$\underset{\sim}{d}^{(2)}$
1 -1
$\underset{\sim}{c}^{(2)}$
0 2
$\mathcal{G}$ $\mathcal{H}$
$\underset{\sim}{d}^{(3)}$
$-\sqrt{2}$
$\underset{\sim}{c}^{(3)}$
$\sqrt{2}$


c(3) √2 H* 1 1
d(3) -√2 G* -1 1
+
0 2
c(2) 0 2 H* 0 0 2/√2 2/√2
d(2) 1 -1 G* 1/√2 -1/√2 -1/√2 1/√2
+
1/√2 -1/√2 1/√2 3/√2
c(1) 1/√2 -1/√2 1/√2 3/√2 H* 1/2 1/2 −1/2 -1/2 1/2 1/2 3/2 3/2
d(1) 1/√2 -5/√2 1/√2 -1/√2 G* 1/2 -1/2 -5/2 5/2 1/2 -1/2 1/2 1/2
+
1 0 -3 2 1 0 1 2

$$(\mathcal{H}a)_k = \sum_n h_{n-2k} a_n$$

$$(\mathcal{G}a)_k = \sum_n g_{n-2k} a_n.$$

$$\underset{\sim}{y} \longrightarrow (\mathcal{G}\underset{\sim}{y}, \mathcal{G}\mathcal{H}\underset{\sim}{y}, \mathcal{G}\mathcal{H}^2\underset{\sim}{y}, \ldots, \mathcal{G}\mathcal{H}^{k-1}\underset{\sim}{y}, \mathcal{H}^k\underset{\sim}{y}).$$

$$(\mathcal{H}^* a)_n = \sum_k h_{n-2k} a_k$$

$$(\mathcal{G}^* a)_n = \sum_k g_{n-2k} a_k.$$

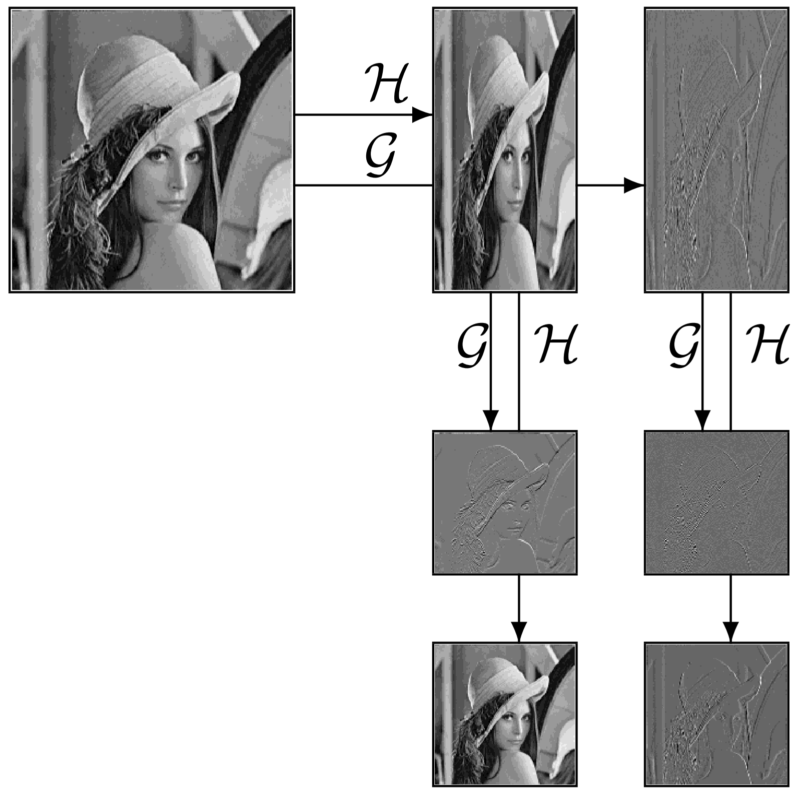
Any function $f \in L_2(R^d)$ can be represented as

$$\begin{aligned} f(x_1,\ldots,x_d) &= \sum_{\mathbf{k}} c_{j_0;\mathbf{k}}\phi_{j_0;\mathbf{k}}(x_1,\ldots,x_d) \\ &+ \sum_{j\geq j_0}\sum_{\mathbf{k}}\sum_{l=1}^{2^d-1} d^{(l)}_{j;\mathbf{k}}\psi^{(l)}_{j;\mathbf{k}}(x_1,\ldots,x_d), \end{aligned}$$

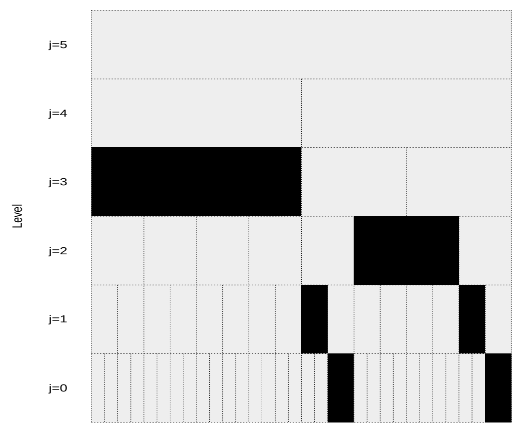
where $\mathbf{k}=(k_1,\ldots,k_d)\in Z^d$ and

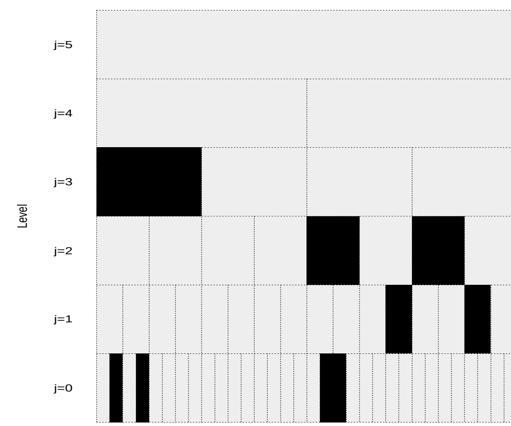
$$\begin{aligned} \phi_{j_0;\mathbf{k}}(x_1,\ldots,x_d) &= 2^{jd/2}\prod_{i=1}^{d}\phi_{(i)}(2^j x_i - k_i) \\ \psi^{(l)}_{j_0;\mathbf{k}}(x_1,\ldots,x_d) &= 2^{jd/2}\prod_{i=1}^{d}\xi_{(i)}(2^j x_i - k_i) \end{aligned}$$

with $\xi=\phi$ or ψ, but not all $\xi=\phi$.



$\mathcal{H}$
$\mathcal{G}$
$\mathcal{G}$
$\mathcal{H}$
$\mathcal{G}$
$\mathcal{H}$





$$\begin{aligned}\int \mathcal{W}_0(x)\,dx &= 1,\\ \mathcal{W}_{2n}(x) &= \sum_k h_k\,\sqrt{2}\,\mathcal{W}_n(2x-k),\\ \mathcal{W}_{2n+1}(x) &= \sum_k g_k\,\sqrt{2}\,\mathcal{W}_n(2x-k),\quad n=0,1,2,\ldots\ .\end{aligned}$$

$$\boxed{\mathcal{W}_{j,n,k}(x) = 2^{j/2}\mathcal{W}_n(2^j x - k),\ (j,n,k)\in Z\times N\times Z.}$$

j – scaling, n – oscillation (sequency), k – translation.

`http://www.isye.gatech.edu/~brani/wavelet.html`

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NEXT: Statistical Modeling in the Wavelet Domain