CNR-IMATI Milano
December 14, 2004

WAVELETS AND SELF-SIMILARITY: THEORY AND APPLICATIONS

Lecture 1: Basics of Wavelets
PLAN

1. What are Wavelets
2. Why Wavelets
3. Mathematics of Wavelets
4. Discrete Wavelet Transform
5. 2-D Case and Wavelet Packets
What are wavelets?

Jean Baptiste Joseph Fourier 1768-1830 and Alfred Haar 1885-1933

The first “wavelet basis” was discovered in 1910 when Alfred Haar showed that any continuous function $f(x)$ on $[0, 1]$ can be approximated by

$$f_n(x) = \langle \xi_0, f \rangle \xi_0(x) + \langle \xi_1, f \rangle \xi_1(x) + \cdots + \langle \xi_n, f \rangle \xi_n(x).$$
The Haar basis is very simple:

\[
\begin{align*}
\xi_0(x) & = 1(0 \leq x \leq 1), \\
\xi_1(x) & = 1(0 \leq x \leq 1/2) - 1(1/2 \leq x \leq 1), \\
\xi_2(x) & = \sqrt{2}[1(0 \leq x \leq 1/4) - 1(1/4 \leq x \leq 1/2)], \\
\cdots \\
\xi_n(x) & = 2^{j/2}[1(k \cdot 2^{-j} \leq x \leq (k + 1/2) \cdot 2^{-j}) \\
& \quad - 1((k + 1/2) \cdot 2^{-j} \leq x \leq (k + 1) \cdot 2^{-j})], \cdots 
\end{align*}
\]

where \( n \) is decomposed as \( n = 2^j + k, \ j \geq 0, \ 0 \leq k \leq 2^j - 1. \)
\[ \sin(\pi x) + \cos(2\pi x) + 0.6 \cdot 1(x > 0.5) \]
Function
\( f(x) = \sin \pi x + \cos 2\pi x + 0.6 \cdot 1(x > 1/2), \quad 0 \leq x \leq 1, \)
and three different levels of approximation in the Haar basis. Approximations \( f_3, f_{15}, \) and \( f_{63} \) are plotted.

For any \( n > 1 \) the basis function \( \xi_n \) can be expressed as a scale-shift transform of a single function \( \xi_1, \)

\[
\xi_n(x) = \xi_{j,k}(x) = 2^{j/2} \xi_1(2^j \cdot x - k), \quad n = 2^j + k.
\]

• \( \xi_n, \ n \geq 1 \) describe the details.

• \( \xi_0(x) \) is responsible for the “average.”
Functions $\xi_1, \xi_2, \xi_{14},$ and $\xi_{25}$ from the Haar basis on $[0, 1]$. 
• The Haar wavelet decomposition of the function

\[ y(x) = \sqrt{x(1-x)} \sin \frac{2.1\pi}{x + 0.05}, \quad 0 \leq x \leq 1. \]

• The function is known as the \textit{doppler} function (Donoho and Johnstone)

• Implemented as a test function in almost all wavelet software packages.
Coefficients of doppler function in Haar basis presented in levels determined by the length of support of corresponding basis functions, $\xi_n$, $n \geq 0$. 
• Wavelets disbalance the data.
• Wavelets whiten the data.
• Wavelets filter the data.
• Wavelets detect self-similarity in data.
California Earthquakes

- A researcher from the Geology Department at Duke University was interested in the possibility of predicting earthquakes by monitoring water levels.
- Water level measurements from six wells located in California, were taken every hour for approximately six years.
- The goal was to smooth the data, eliminate the noise, and inspect the signal at pre-earthquake time.
Raw data for hourly measurements (one year, $8192 = 2^{13}$ observations). The line-like artifact (enlarged in right panel) corresponds to the earthquake time (Julian day of 417).
Comparison of several smoothing methods. Upper, Left: Data smoothed by kernel method (normal window, k=5); Upper, Right: Data smoothed by `loess` method; Lower, Left: Data smoothed by `supsmu` method; Lower, Right: Wavelet Smoothed Data
Multiresolution analysis (MRA) is a sequence of closed subspaces $V_n$, $n \in \mathbb{Z}$ in $L_2(R)$

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$$

$$\cap_n V_n = \{0\}, \quad \cup_n V_n = L_2(R).$$

The hierarchy in MRA is constructed such that

- $V$-spaces are self similar,

  $$f(x) \in V_0 \text{ iff } f(2^n x) \in V_n.$$
• There exists a scaling function \( \phi \in V_0 \) whose integer translates span \( V_0 \),

\[
V_0 = \{ f \in L_2(R) \mid f(x) = \sum_k c_k \phi(x - k) \},
\]

and for which the set \( \{ \phi(\bullet - k), \ k \in \mathbb{Z} \} \) is an orthonormal basis.

• (Scaling equation) Since \( V_0 \subset V_1 \),

\[
\phi(x) = \sum_{k \in \mathbb{Z}} h_k \sqrt{2} \phi(2x - k).
\]
• **Normalization.**

\[ \sum_{k \in \mathbb{Z}} h_k = \sqrt{2}. \]

• **Orthogonality.** For any \( l \in \mathbb{Z}, \)

\[ \sum_{k} h_k h_{k-2l} = \delta_l. \]

An important special case is \( l = 0 \)

\[ \sum_{k} h_k^2 = 1. \]
Link between $h$ and $\phi$

Transfer function:

$$m_0(\omega) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} h_k e^{-i k \omega} \quad [= \frac{1}{\sqrt{2}} H(\omega)].$$

Recall the scaling equation:

$$\phi(x) = \sum_{k \in \mathbb{Z}} h_k \sqrt{2} \phi(2x - k).$$

In the Fourier domain:

$$\Phi(\omega) = m_0 \left( \frac{\omega}{2} \right) \Phi \left( \frac{\omega}{2} \right).$$
\[ \Phi(\omega) = m_0 \left( \frac{\omega}{2} \right) \Phi \left( \frac{\omega}{2} \right) \]

\[ \Phi(\omega) = \int_{-\infty}^{\infty} \phi(x) e^{-i\omega x} \, dx \]

\[ = \sum_k \sqrt{2} h_k \int_{-\infty}^{\infty} \phi(2x - k) e^{-i\omega x} \, dx \]

\[ = \sum_k \frac{h_k}{\sqrt{2}} e^{-ik\omega/2} \int_{-\infty}^{\infty} \phi(2x - k) e^{-i(2x-k)\omega/2} \, d(2x - k) \]

\[ = \sum_k \frac{h_k}{\sqrt{2}} e^{-ik\omega/2} \Phi \left( \frac{\omega}{2} \right) \]

\[ = m_0 \left( \frac{\omega}{2} \right) \Phi \left( \frac{\omega}{2} \right) \implies \Phi(\omega) = \prod_{n=1}^{\infty} m_0 \left( \frac{\omega}{2^n} \right). \]
Mother Wavelet $\psi$

- Whenever sequence of subspaces satisfy MRA properties there exists an orthonormal basis for $L_2(R)$,

$$\{\psi_{jk}(x) = 2^{j/2} \psi(2^j x - k), \ j, k \in \mathbb{Z}\}$$

such that $\{\psi_{jk}(x), \ j\text{-fixed}, \ k \in \mathbb{Z}\}$ is an orthonormal basis for the “difference space” $W_j = V_{j+1} \ominus V_j$.

(because of containment $W_0 \subset V_1$),

$$\psi(x) = \sum_{k \in \mathbb{Z}} g_k \sqrt{2} \phi(2x - k),$$
• Quadrature mirror relation

\[ g_n = (-1)^n h_{1-n}. \]

• Any \( L_2(R) \) function can be represented as

\[ f(x) = \sum_{j,k} d_{jk} \psi_{jk}(x), \quad [L_2(R) = \bigoplus_{j=-\infty}^{\infty} W_j]. \]

For any fixed \( j_0 \) the decomposition \( L_2(R) = V_{j_0} \oplus \oplus_{j=j_0}^{\infty} W_j \) corresponds to representation

\[ f(x) = \sum_{k} c_{j_0,k} \phi_{j_0,k}(x) + \sum_{j \geq j_0} \sum_{k} d_{jk} \psi_{j,k}(x). \]
Haar Example

\[ \phi(x) = \phi(2x) + \phi(2x - 1) = \sqrt{2/2} \phi(2x) + \sqrt{2/2} \phi(2x - 1), \]

\[ \psi(x) = \phi(2x) - \phi(2x - 1) = \sqrt{2/2} \phi(2x) - \sqrt{2/2} \phi(2x - 1), \]

which gives the wavelet filter coefficients:

\[ h_0 = h_1 = \frac{1}{\sqrt{2}}, \quad \text{and} \]

\[ g_0 = -g_1 = \frac{1}{\sqrt{2}}. \]
Left: Multiresolution Analysis of doppler function. The original function in $V_{10}$ is a sum of projections on $V_4$ and $W_4$–$W_9$ subspaces. Right: Coarsening doppler function by projecting it to $V$-subspaces.
\[ h_0 = h_1 = \frac{1}{\sqrt{2}}. \]

For the Haar wavelet, the transfer function becomes

\[
m_0(\omega) = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} e^{-i\omega_0} \right) + \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} e^{-i\omega_1} \right) = \frac{1 + e^{-i\omega}}{2}.
\]

\[
m_1(\omega) = -e^{-i\omega} m_0(\omega + \pi) = -e^{-i\omega} \left( \frac{1}{2} - \frac{1}{2} e^{i\omega} \right) = \frac{1 - e^{-i\omega}}{2}.
\]
Functions $|m_0(\omega)|^2 = \cos^2 \frac{\omega}{2}$ and $|m_1(\omega)|^2 = \sin^2 \frac{\omega}{2}$ for the Haar case.
Ingrid Daubechies was first to construct compactly supported orthogonal wavelets with preassigned degree of smoothness.

Graphs of scaling and wavelets functions from Daubechies family, $N = 2, 3, 4, 8, \text{ and } 10$. 
<table>
<thead>
<tr>
<th>$k$</th>
<th>DAUB2</th>
<th>DAUB3</th>
<th>DAUB4</th>
<th>DAUB5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.4829629131445</td>
<td>0.3326705529500</td>
<td>0.2303778133088</td>
<td>0.1601023979741</td>
</tr>
<tr>
<td>1</td>
<td>0.8365163037378</td>
<td>0.8068915093110</td>
<td>0.7148465705529</td>
<td>0.6038292697971</td>
</tr>
<tr>
<td>2</td>
<td>0.224143680420</td>
<td>0.4598775021184</td>
<td>0.6308807679298</td>
<td>0.7243085284377</td>
</tr>
<tr>
<td>3</td>
<td>-0.1294095225512</td>
<td>-0.1350110200102</td>
<td>-0.0279837694168</td>
<td>0.1384281459013</td>
</tr>
<tr>
<td>4</td>
<td>-0.1294095225512</td>
<td>-0.0854412738820</td>
<td>-0.1870348117190</td>
<td>-0.2422948870663</td>
</tr>
<tr>
<td>5</td>
<td>0.0352262918857</td>
<td>0.0308413818355</td>
<td>0.0308413818355</td>
<td>-0.032448695846</td>
</tr>
<tr>
<td>6</td>
<td>0.0328830116668</td>
<td>0.0328830116668</td>
<td>0.0328830116668</td>
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<tr>
<td>7</td>
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<tr>
<td>8</td>
<td>-0.0125807519990</td>
<td>-0.0125807519990</td>
<td>-0.0125807519990</td>
<td>0.0033357252854</td>
</tr>
</tbody>
</table>
\[
\begin{align*}
\begin{cases}
\ h_0 + h_1 + h_2 + h_3 &= \sqrt{2} \\
\ h_0^2 + h_1^2 + h_2^2 + h_3^2 &= 1 \\
\ -h_1 + 2h_2 - 3h_3 &= 0 \\
\ h_0 \ h_2 + h_1 \ h_3 &= 0
\end{cases}
\end{align*}
\]

- Pollen-type Parameterization \([s = 2\sqrt{2}]\):

<table>
<thead>
<tr>
<th>(n)</th>
<th>(h_n) for (N = 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>((1 + \cos \varphi - \sin \varphi)/s)</td>
</tr>
<tr>
<td>1</td>
<td>((1 + \cos \varphi + \sin \varphi)/s)</td>
</tr>
<tr>
<td>2</td>
<td>((1 - \cos \varphi + \sin \varphi)/s)</td>
</tr>
<tr>
<td>3</td>
<td>((1 - \cos \varphi - \sin \varphi)/s)</td>
</tr>
</tbody>
</table>
- Strang - Fix condition \[ \sum_k \phi(x - k) = 1, \]
  \[ \sum_k (x - \text{Const}) \phi(x - k) = x \], ...
## Discrete Wavelet Transforms

<table>
<thead>
<tr>
<th>Fourier Methods</th>
<th>Fourier Integrals</th>
<th>Fourier Series</th>
<th>Discrete Fourier Transforms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wavelet Methods</td>
<td>Continuous</td>
<td>Wavelet Series</td>
<td>Discrete Wavelet Transforms</td>
</tr>
<tr>
<td></td>
<td>Wavelet Transforms</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
• The Cascade Algorithm [Mallat, 1989]

\[ H : \quad c_{j+1,l} = \sum_{k} h_{k-2l} c_{j,k} \]

\[ G : \quad d_{j+1,l} = \sum_{k} g_{k-2l} c_{j,k} \]

\[ H^*, G^* : \quad c_{j,k} = \sum_{l} c_{j+1,l} h_{k-2l} + \sum_{l} d_{j+1,l} g_{k-2l}. \]
\[ \cdots \xrightarrow{d^{(3)}} \oplus \xrightarrow{d^{(2)}} \oplus \xrightarrow{d^{(1)}} \oplus \xrightarrow{G^* H^*} \cdots \]
Let $\tilde{y} = (1, 0, -3, 2, 1, 0, 1, 2)$. The values $f(n) = y_n$, $n = 0, 1, \ldots, 7$ are interpolated by piecewise constant function. Assume that $f$ belongs to Haar’s multiresolution space $V_0$.

$h = \{h_0, h_1\} = \{1/\sqrt{2}, 1/\sqrt{2}\}$; $g = \{g_0, g_1\} = \{1/\sqrt{2}, -1/\sqrt{2}\}$. 
\[ y = c^{(0)} \]
\[
\begin{array}{cccccc}
1 & 0 & -3 & 2 & 1 & 0 & 1 & 2
\end{array}
\]

\[ d^{(1)} \]
\[
\begin{array}{ccc}
\frac{1}{\sqrt{2}} & -\frac{5}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{array}
\]

\[ c^{(1)} \]
\[
\begin{array}{ccc}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}}
\end{array}
\]

\[ d^{(2)} \]
\[
\begin{array}{c}
1 & -1
\end{array}
\]

\[ c^{(2)} \]
\[
\begin{array}{c}
0 & 2
\end{array}
\]

\[ d^{(3)} \]
\[
\begin{array}{c}
-\sqrt{2}
\end{array}
\]

\[ c^{(3)} \]
\[
\begin{array}{c}
\sqrt{2}
\end{array}
\]
\[(H a)_k = \sum_n h_{n-2k} a_n \]
\[(G a)_k = \sum_n g_{n-2k} a_n.\]

\[y \longrightarrow (G y, GHy, GH^2y, \ldots, GH^{k-1}y, H^k y).\]

\[(H^* a)_n = \sum_k h_{n-2k} a_k \]
\[(G^* a)_n = \sum_k g_{n-2k} a_k.\]
Any function \( f \in L_2(R^d) \) can be represented as

\[
f(x_1, \ldots, x_d) = \sum_{\mathbf{k}} c_{j_0; \mathbf{k}} \phi_{j_0; \mathbf{k}}(x_1, \ldots, x_d) \\
+ \sum_{j \geq j_0} \sum_{\mathbf{k}} \sum_{l=1}^{2^d-1} d_{j; \mathbf{k}}^{(l)} \psi_{j; \mathbf{k}}^{(l)}(x_1, \ldots, x_d),
\]

where \( \mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d \) and

\[
\phi_{j_0; \mathbf{k}}(x_1, \ldots, x_d) = 2^{jd/2} \prod_{i=1}^{d} \phi(i)(2^j x_i - k_i)
\]

\[
\psi_{j_0; \mathbf{k}}^{(l)}(x_1, \ldots, x_d) = 2^{jd/2} \prod_{i=1}^{d} \xi(i)(2^j x_i - k_i)
\]

with \( \xi = \phi \) or \( \psi \), but not all \( \xi = \phi \).
\[
\int W_0(x) \, dx = 1,
\]
\[
W_{2n}(x) = \sum_k h_k \sqrt{2} W_n(2x - k),
\]
\[
W_{2n+1}(x) = \sum_k g_k \sqrt{2} W_n(2x - k), \quad n = 0, 1, 2, \ldots.
\]
\[
W_{j,n,k}(x) = 2^{j/2} W_n(2^j x - k), \quad (j,n,k) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{Z}.
\]

\(j\) – scaling, \(n\) – oscillation (sequency), \(k\) – translation.


Software Wavelab (Donoho and coauthors, Stanford)
NEXT: Statistical Modeling in the Wavelet Domain