Optimal control of culling in epidemic models for wildlife

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Introduction

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1. Reactive culling (Donnelly et al., Nature, 2006): intensified hunting campaigns at the onset of the epidemics (bovine tuberculosis in British badger, rabies in European fox and Canadian raccoon)
2. Epidemic-transient phase culling (Schnyder et al., Veter. Rec. 2002): targeted hunting campaigns at the end of the first epidemic peak (classic swine fever in Swiss wild boar)

Research question

When is reactive culling the optimal strategy?

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$\Rightarrow$ Optimal control theory applied to a SIR Model
with the objective to minimize
both the number of infected animals
and the cost of culling effort

Mathematical model

$$\begin{cases} \dot{S} = r\,S\left(1 - \dfrac{S+I+R}{K}\right) + \nu R - \beta S I - c(t)\,S, \\ \dot{I} = \beta S I - (\alpha + \mu + \eta + c(t))\,I, \\ \dot{R} = \eta I - (\mu + c(t))\,R \end{cases} \tag{1}$$

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Objective functional to minimize:

$$J(c) = \int_0^T \left(I(t)^\gamma + P c(t)^\theta \right) dt, \qquad \gamma,\ \theta \in \{1, 2\}$$

in the class of admissible control

$$U = \left\{ c(t) \text{ piecewise continuous} \mid 0 \le c(t) \le c_{max},\ \forall\ t \in [0, T] \right\} \tag{2}$$

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define the Hamiltonian function

$$\mathcal{H}(\lambda, x, c) = \lambda \cdot \mathbf{f}(x, c) = \sum_{j=0}^{n} \lambda_j f^j(x, c)$$

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- Solve these equations with boundary conditions (initial and final, respectively)
- Explicit expression for the optimal control

SI model – Quadratic costs

SI epidemic model (in the limit of vanishing removal rate)

$$\begin{cases} \dot{S} = r\,S\left(1 - \dfrac{S+I}{K}\right) - \beta SI - c\,S, \\ \dot{I} = \beta SI - (\alpha + \mu + c)\,I, \end{cases}$$

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Optimal control

$$c^* = \max\Big(0, \min\big(\hat{c}, c_{max}\big)\Big), \quad \text{where} \quad \hat{c} = \frac{\lambda_1(t)S(t) + \lambda_2(t)I(t)}{2P}.$$

Linear costs $J(c) = \int \left[I(t) + Pc(t) \right] dt$

Optimal control

$$c^*(t) = \begin{cases} 0 & \text{if } \psi(t) > 0 \\ c_{sing} & \text{if } \psi(t) = 0 \\ c_{max} & \text{if } \psi(t) < 0\,, \end{cases}$$

with $\psi(t) = \dfrac{\partial \mathcal{H}}{\partial c} = P - \lambda_1 S - \lambda_2 I$ *switching function*;

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Generalized Legendre Clebsch condition (for the singular control to be optimal)

$$(-1)^2 \frac{\partial}{\partial c} \left[\frac{d^4}{dt^4} \frac{\partial \mathcal{H}}{\partial c} \right] = Q \geq 0$$

$$Q(S,I) = \frac{(P\beta - 1)SI}{S+I} (\alpha + \mu + r) \left[\beta S - \left(\frac{r}{K} + \beta \right) I \right].$$

Optimal control for fast epidemics

$$\begin{cases} \dot{S} = -\beta S I - c\, S, \\ \dot{I} = \beta S I - (\alpha + c)\, I, \end{cases}$$

Theoretical results for fast epidemic model with linear costs

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The optimal control c^* is bang-bang,

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Theorem

Epidemic-transient phase culling cannot occur

$$\psi(0) = \frac{\partial \mathcal{H}}{\partial c}\bigg|_{t=0} > 0 \quad \Rightarrow \quad \psi(t) = \frac{\partial \mathcal{H}}{\partial c} > 0\,, \quad \forall\, t\,.$$

Optimal culling: numerical simulations – quadratic costs

Numerical discretization: Forward-Backward Sweep method (Lenhart 2007)

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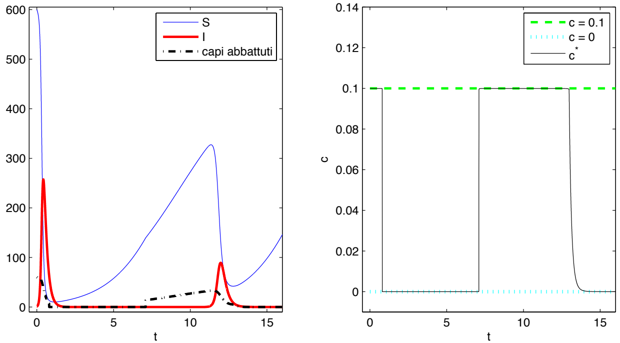
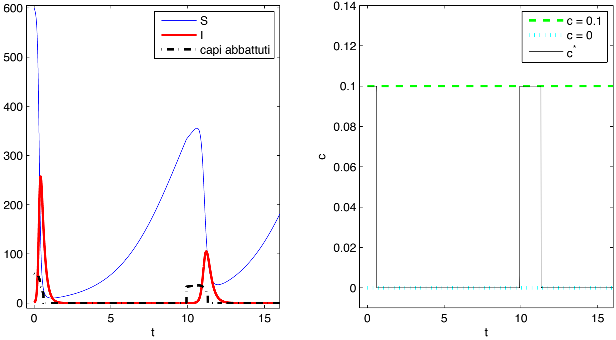


Figure: $T = 16$ years. Initial conditions $S(0) = K$, $I(0) = 1$.

Optimal culling – linear costs



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Sensitivity analysis – linear costs

The optimal control depends on the parameters of the model

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- Extensive numerical study by varying R_0, culling cost P, virulence parameters α, β

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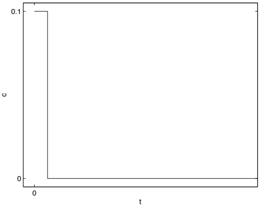
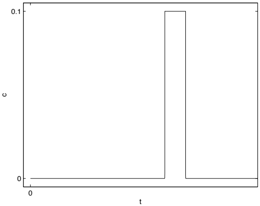
- Extensive numerical study by varying R_0, culling cost P, virulence parameters α, β
- Classification of results according to reactive culling ($\bullet$), epidemic-transient phase culling ($\triangle$), or no culling ($*$)

($\bullet$)



($\triangle$)



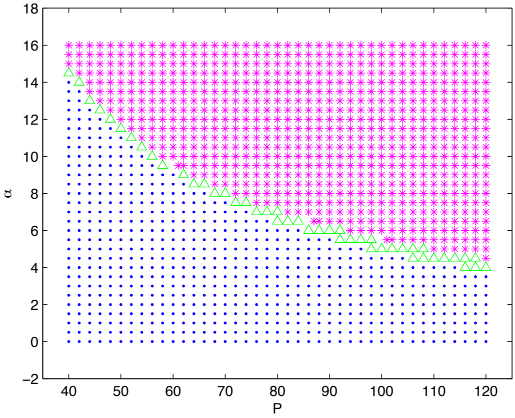


Figure: $T = 6$ years. α ($0 \leq \alpha \leq 16$, step 0.5), P ($40 \leq P \leq 120$, step 2) and $\beta = R_0 \frac{\alpha+\mu}{K}$ with $R_0 = 4.6154$.

Numerical results SI – Variation of R_0 and P

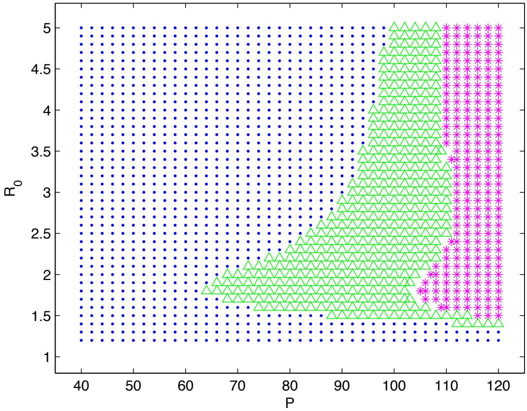


Figure: $T = 5$ years. P ($40 \leq P \leq 120$ step 2), R_0 ($1.2 \leq R_0 \leq 5$, step 0.1) and $\beta = R_0 \frac{\alpha+\mu}{K}$.

Effect of immunity on optimal culling

SIR epidemic model

$$\begin{cases} \dot{S} = r\,S\left(1 - \dfrac{S+I+R}{K}\right) + \nu R - \beta SI - c(t)\,S\,, \\ \dot{I} = \beta SI - (\alpha + \mu + \eta + c(t))\,I\,, \\ \dot{R} = \eta I - (\mu + c(t))\,R\,, \\ S(0) = K,\ I(0) = 1,\ R(0) = 0\,. \end{cases}$$

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Objective cost function to minimize

$$J(c) = \int_0^T \Big(I(t) + P c(t)\Big)\,dt\,, \qquad 0 \le c(t) \le c_{max}.$$

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Hamiltonian functional:

$$\begin{aligned} \mathcal{H} =& I + Pc + \lambda_1 \nu (S+R) - \lambda_1 \frac{r}{K}\Big(S+I+R\Big)S - \lambda_1(\mu + c)S \\ & - \lambda_1 \beta SI + \lambda_2 \beta SI - \lambda_2(\alpha + \mu + \eta + c)\,I + \lambda_3 \eta I - \lambda_3(\mu + c)\,R\,. \end{aligned}$$

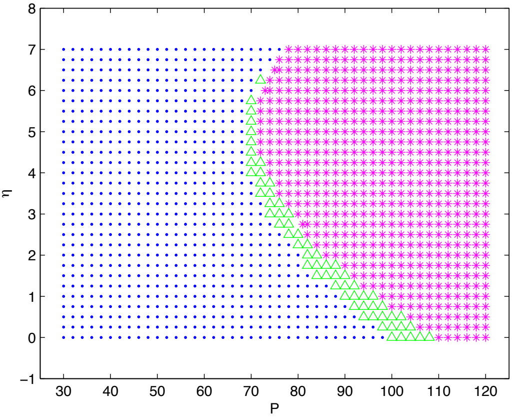

Figure: $T = 1.5$ years. P ($30 \leq P \leq 120$, step 2), η ($0 \leq \eta \leq 7$, step 0.25) and $\beta = R_0 \frac{\alpha+\mu+\eta}{K}$ with $R_0 = 4.6154$.

Numerical results SIR – Variation of R_0 and P

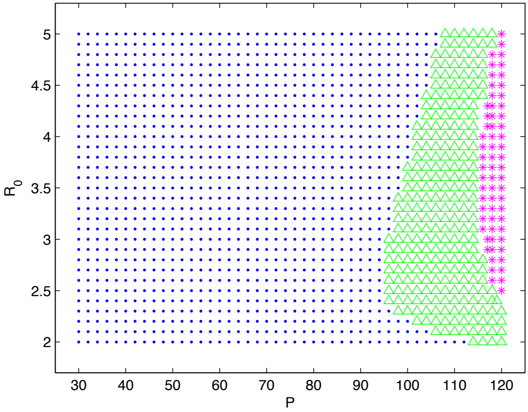

Figure: $T = 2.5$ years. P ($30 \leq P \leq 120$, step 2), R_0 ($2 \leq R_0 \leq 5$ step 0.1) and $\beta = R_0 \frac{\alpha+\mu+\eta}{K}$ ($\eta = 1,\ \alpha = 4$).

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References

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